On the propagation of jump discontinuities in relativistic cosmology

Henk van Elst,[1*] George F. R. Ellis,[1†] and Bernd G. Schmidt[2‡]
[1]*Cosmology Group, Department of Mathematics and Applied Mathematics, University of Cape Town Rondebosch 7701, Cape Town, South Africa*
[2]*Albert–Einstein–Institut, Max–Planck–Institut für Gravitationsphysik, Am Mühlenberg 1 14476 Golm, Germany*
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A recent dynamical formulation at derivative level $\partial^3 g$ for fluid spacetime geometries $(\mathcal{M}, \mathbf{g}, \mathbf{u})$, that employs the concept of evolution systems in first-order symmetric hyperbolic format, implies the existence in the Weyl curvature branch of a set of timelike characteristic 3-surfaces associated with propagation speed $|\,v\,| = \frac{1}{2}$ relative to fluid-comoving observers. We show it is the physical role of the constraint equations to prevent realisation of jump discontinuities in the derivatives of the related initial data so that Weyl curvature modes propagating along these 3-surfaces cannot be activated. In addition we introduce a new, illustrative first-order symmetric hyperbolic evolution system at derivative level $\partial^2 g$ for baryotropic perfect fluid cosmological models that are invariant under the transformations of an Abelian G_2 isometry group.

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I. INTRODUCTION

It is well known that the mathematical nature of the evolution system within the relativistic gravitational field equations is essentially *hyperbolic*, with domains of dependence and influence determined by the speed of light. However, this nature is not obvious when the dynamical equations are written out in standard form, employing either a metric approach [21], a Hamiltonian (ADM) representation [1], a $1+3$ orthonormal frame (ONF) formulation [5,19], or a form obtained from a covariant $(1+3)$–decomposition [3,6]. Consequently, over the years considerable effort has gone into determining ways of making this hyperbolic nature clear. Indeed, evolution systems of partial differential equations in *first-order symmetric hyperbolic* (FOSH) format have proven to be a main theme of research for at least the last five years. This activity was particularly motivated by the desire to carry over to the numerical investigation of relativistic effects such as generation of gravitational radiation associated with the in-spiralling of black hole and neutron star binaries the methods and expertise gained in areas of computational physics with a longer tradition like, e.g., hydrodynamics, where evolution systems of FOSH format are commonplace (see, e.g., the recent reviews [24] and [13]).

One of the most promising methods for obtaining a FOSH representation for the relativistic gravitational field equations is to use an extended $1+3$ orthonormal frame formulation that includes the once-contracted second Bianchi identities [10]. In terms of the highest-order partial derivatives of the spacetime metric $\mathbf{g}$ implicitly occurring, the dynamical equations here rank at derivative level $\partial^3 g$. The set of geometrically defined field variables contains the components of the physically significant Weyl curvature tensor, and so we refer to this as a $1+3$ orthonormal frame *curvature* formulation of gravitational fields. It represents a completion of the physically and geometrically transparent $1+3$ covariant formalism for the cosmological case, in which all field variables are covariantly defined relative to a uniquely defined future-directed timelike reference congruence given by the unit 4-velocity field $\mathbf{u}$ of matter present [6,7]. As Friedrich first showed, when the matter source for a spacetime geometry $(\mathcal{M}, \mathbf{g}, \mathbf{u})$ is described phenomenologically as a perfect fluid, or one restricts to vacuum situations, upon introduction of a set of local coordinates and a specific choice to remove the gauge fixing freedom it is indeed possible to find linear combinations of the field variables and their dynamical equations that lead to an evolution system of (autonomous) partial differential equations in FOSH format, as desired [11,12]. It is puzzling, then, that when this is done, as well as the expected sets of characteristic 3-surfaces associated with propagation speeds relative to $\mathbf{u}$ of $|\,v\,| = 0$ (Coulomb-like gravitational and fluid rotational modes), $|\,v\,| = c_s$ (sound wave modes), and $|\,v\,| = 1$ (transverse gravitational wave

*Electronic address: henk@gmunu.mth.uct.ac.za
†Electronic address: ellis@maths.uct.ac.za
‡Electronic address: bernd@aei-potsdam.mpg.de

modes), the equations on the face of it indicate that the semi-longitudinal Weyl curvature characteristic eigenfields, identified long ago by Szekeres [26], appear related to an additional set of timelike characteristic 3-surfaces associated with propagation speed relative to $\mathbf{u}$ of one-half the speed of light, $|\,v\,| = \frac{1}{2}$.

Friedrich assumed that these modes could be ignored, because, at least in the *vacuum case*, they had no invariant meaning [11]. Two of us (HvE and GFRE), on the other hand, argued in a recent paper that in the fluid context that is relevant in relativistic cosmology, this is not the case, essentially because (i) the timelike reference congruence corresponding to $\mathbf{u}$ is invariantly defined by the presence of matter, and (ii) the hypothetical Weyl curvature modes cannot all be transformed to zero by using up the remaining gauge fixing freedom [9]. Thus, these might possibly represent propagation modes that are physically meaningful. In private communications, Friedrich responded that the form of the $\partial^3 g$-order FOSH evolution systems discussed in Refs. [12] and [9] is *not* unique; one could obtain other reductions of the relativistic gravitational field equations that would lead to additional timelike characteristic 3-surfaces associated with propagation speeds *different* from $|\,v\,| = \frac{1}{2}$. And, hence, these modes cannot be physical (see also Ref. [13]). However, that argument is not very satisfactory. In our view, it compounds the problem, rather than solving it, by suggesting (if we take for face value the FOSH formalism and its interpretation) that still other (subluminal) propagation speeds could be associated with the transport of physical quantities, as well as $|\,v\,| = \frac{1}{2}$. The implication is that any single FOSH evolution system cannot be taken seriously without considering the set of *all possible* FOSH evolution systems, which is difficult to determine.

What had not been fully taken into account in these discussions, even though Sec. III D 2 of Ref. [9] made a brief reference to this aspect, is the physical role the *constraint equations* play in the selection of appropriate classes of initial data sets. Specifically, what kind of physically relevant *jump discontinuities* they do allow to be present in the derivatives of these data sets [23]. Jump discontinuities in the derivatives of the initial data are of major dynamical importance as they can be thought of as representing the triggering or terminating events of physical generation processes.[1] In the present paper we show that an analysis of the constraint equations leads to a resolution of the problem: *while the additional timelike characteristic 3-surfaces do indeed occur in the $\partial^3 g$-order FOSH evolution system, the constraint equations are of such a form as to prevent the corresponding semi-longitudinal Weyl curvature modes being activated*. Because jump discontinuities *cannot* exist in the values of the constraint equations themselves (they have to be zero everywhere), jump discontinuities in the derivatives of the initial data can only be propagated along the characteristic 3-surfaces associated with propagation speeds relative to $\mathbf{u}$ of $|\,v\,| = 0$, $|\,v\,| = c_s$, and $|\,v\,| = 1$, but not along those associated with $|\,v\,| = \frac{1}{2}$. Hence, one cannot physically send arbitrary information along the latter set of timelike characteristic 3-surfaces.

The problem of additional characteristic 3-surfaces and how the constraint equations select appropriate initial data sets can be highlighted in a nice and transparent fashion by a suggestive example that derives from a set of linearised relativistic gravitational field equations given in a paper by Kind, Ehlers and Schmidt [17]. We will briefly discuss this example in Appendix A 1.

A possible complementary explanation to the above viewpoint arises from the following chain of arguments:

(i) For vacuum as well as perfect fluid spacetime geometries, there exist $\partial^2 g$-order dynamical formulations of the relativistic gravitational field equations (with harmonic coordinate gauge fixing) in which characteristic 3-surfaces *only* correspond to the local light cones, the local sound cones, or are generated by the fluid flow lines. (See, e.g., Refs. [24] and [13].)

(ii) There are dynamical formulations of order $\partial^2 g$ or $\partial^3 g$ that have *additional* (timelike or spacelike) characteristic 3-surfaces. Along these characteristic 3-surfaces the evolution system will propagate jump discontinuities if we put them into the related initial data (*ignoring* the constraint equations).

(iii) Different dynamical formulations with different sets of characteristic 3-surfaces all describe for *identical* initial data the *same* spacetime geometry.

(iv) *Because* of the existence of a dynamical formulation according to (i), jump discontinuities as described in (ii) *cannot* really exists (and so be physically meaningful). Hence, the only logical possibility is that in (ii) the constraint equations prevent the existence of such jump discontinuities.

The present paper shows that the semi-longitudinal Weyl curvature modes that potentially exist cannot in fact carry "gravitational news". Of course, this is expected, so this analysis simply confirms what everyone has believed all along. However, we believe it is nevertheless useful, both in terms of showing important relations between the constraint equations and the set of characteristic 3-surfaces that can occur in general FOSH evolution systems (and have not been clearly demonstrated in the literature), and because this does indeed resolve the issue at hand in the specific important case of relativistic gravitation, where the variables presented here, and hence the associated dynamical

[1] Jürgen Ehlers (private communication).

equations, have much to recommend them. Without the analysis given in this paper, the apparent $(|\,v\,| = \frac{1}{2})$–modes remain an annoying and unresolved problem.

The outline of the paper is as follows. In Sec. II we briefly discuss the mathematical concepts relevant to our analysis: we review the central ideas behind FOSH evolution systems, we address the issue of gauge fixing freedom in a $1+3$ orthonormal frame formulation of the relativistic gravitational field equations, we list the set of constraint equations we need to consider, and we introduce a $(1+1+2)$–decomposition of all geometrically defined field variables. Then, in Sec. III, we address from a purely *local* viewpoint the question of how the constraint equations select appropriate initial data sets for the $\partial^3 g$-order FOSH evolution system introduced in Ref. [9]. To the best of our knowledge, we present in Sec. IV for the first time a fully gauge-fixed autonomous FOSH evolution system for a special class of spatially inhomogeneous perfect fluid cosmological models on the basis of a $1+3$ orthonormal frame formulation that is only of order $\partial^2 g$ in the degrees of freedom of the gravitational field.[2] We refer to this as a $1+3$ orthonormal frame *connection* formulation of gravitational fields. Additionally, we derive for these cosmological models, for which there exists an Abelian G_2 isometry group, the transport equations that describe how physically relevant jump discontinuities in the derivatives of the initial data are propagated along the bicharacteristic rays of the setting. Our conclusions are contained in Sec. V. Finally, besides the above-mentioned suggestive example in Appendix A 1, we give in Appendix A 2 in explicit form those linear combinations of the components of the extended $1+3$ orthonormal frame constraint equations that prove suitable for our analysis.

We will use the same conventions, units and notations as introduced in Appendix A 1 of Ref. [9].

II. MATHEMATICAL PRELIMINARIES

A. First-order symmetric hyperbolic evolution systems

We consider evolution systems for a collection of k real-valued field variables $u^A = u^A(x^\mu)$ that are composed of a set of k quasi-linear partial differential equations of *first order* given by

$$M^{AB\,\mu}(x^\nu, u^C)\,\partial_\mu u_B = N^A(x^\nu, u^C)\ , \qquad A,\,B,\,C = 1,\ldots,k\ ; \tag{2.1}$$

the field variables u^A are functions of a set of local spacetime coordinates $\{\,x^\mu\,\}$. Evolution systems of this form are called *symmetric* if the real-valued $k \times k$ coefficient matrices entering the principle part satisfy $M^{AB\,\mu} = M^{(AB)\,\mu}$; moreover, they are called *hyperbolic* if the contraction $M^{AB\,\mu}\,n_\mu$ with the coordinate components of an arbitrary *past-directed* timelike 1-form n_a yields a positive-definite matrix. We remark that (i) cases with $M^{AB\,\mu} = M^{AB\,\mu}(x^\nu)$ are referred to as *semi-linear*, and (ii) cases with $M^{AB\,\mu} = M^{AB\,\mu}(u^C)$ and $N^A = N^A(u^C)$ are referred to as *autonomous*. In general, it proves convenient to consider a $(1+3)$–decomposition of Eq. (2.1) in the format

$$M^{AB\,0}(t, x^j, u^C)\,\partial_t u_B + M^{AB\,i}(t, x^j, u^C)\,\partial_i u_B = N^A(t, x^j, u^C)\ .$$

The concept of FOSH evolution systems was first introduced by Friedrichs [14]; a standard reference, broughtly discussing its usefulness to applications in mathematical physics and also presenting proofs for existence and uniqueness of solutions, is the book by Courant and Hilbert [2]. The *characteristic condition*

$$0 = Q := \det\,[\,M^{AB\,\mu}\,\nabla_\mu\phi\,] \tag{2.2}$$

determines the coordinate components of the *past-directed* normals $\nabla_a\phi$ of the set of *characteristic 3-surfaces* $\mathcal{C}$:$\{\phi(x^\mu) = \text{const}\}$ associated with the FOSH evolution system (2.1). With $M^{AB\,\mu} = M^{(AB)\,\mu}$, hyperbolicity of Eq. (2.1) thus also corresponds to all individual roots ("eigenvalues") v of Eq. (2.2) being *real-valued*. Every individual v then defines a pair of so-called *left* and *right nullifying vectors*, l^A and r^A, by

$$0 =: l_A\,(M^{AB\,\mu}\,\nabla_\mu\phi)\ , \qquad 0 =: (M^{AB\,\mu}\,\nabla_\mu\phi)\,r_B\ ; \tag{2.3}$$

the linearly independent sets $\{\,l^A\,\}$ or $\{\,r^A\,\}$ form a basis of the k-dimensional space of field variables u^A.

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[2]The equations of motion for the matter sources will necessarily derive from the twice-contracted second Bianchi identities and so are typically of order $\partial^3 g$.

According to the theory discussed in Ch. VI.4.2 of Courant and Hilbert [2], FOSH evolution systems of the format (2.1) have the power to describe the physical transport along so-called *bicharacteristic rays* of *jump discontinuities* that exist in the *outward first derivatives* across a characteristic 3-surface $\mathcal{C}:\{\phi(x^\mu) = \text{const}\}$ of the field variables u^A; the tangential first derivatives of the u^A as well as the u^A themselves are assumed to be *continuous* across $\mathcal{C}:\{\phi(x^\mu) = \text{const}\}$. As is standard, we will use the notation

$$[f] := \lim_{\phi\to c_+} f - \lim_{\phi\to c_-} f = f_+ - f_-$$

to symbolise a jump discontinuity (of finite magnitude) across $\mathcal{C}:\{\phi(x^\mu) = \text{const}\}$ in the value of a given variable f. Under the stated assumptions, it follows from Eqs. (2.1) and (2.3) that

$$0 = (M^{AB\,\mu}\,\nabla_\mu\phi)\,[\,\partial_\phi u_B\,] \qquad \Leftrightarrow \qquad [\,\partial_\phi u^A\,] = [\partial_\phi u]\,r^A \ , \tag{2.4}$$

i.e., the jump discontinuity $[\,\partial_\phi u^A\,]$ must be proportional to a right nullifying vector r^A. The real-valued scalar of proportionality, denoted by $[\partial_\phi u]$, is assumed to have continuous first derivatives. Then, according to Chs. VI.4.2 and VI.4.9 of Ref. [2], for linear, semi-linear and quasi-linear FOSH evolution systems (2.1) the *transport equation* for $[\partial_\phi u]$ along bicharacteristic rays within the characteristic 3-surfaces $\mathcal{C}:\{\phi(x^\mu) = \text{const}\}$ takes the effective form

$$0 = (l_A M^{AB\,\mu} r_B)\,\partial_\mu[\partial_\phi u] + \big(\,(l_A M^{AB\,\mu})\,\partial_\mu r_B - (l_A N^{AB} r_B)\,\big)\ [\partial_\phi u] \ . \tag{2.5}$$

Note, in particular, the *involutive character* of this relation; if $[\partial_\phi u]$ is non-zero at one point along a bicharacteristic ray, it will be non-zero everywhere along this ray, and vice versa. Note also that the present treatment of jump discontinuities breaks down when the $\mathcal{C}:\{\phi(x^\mu) = \text{const}\}$ within a given family start to intersect and so prompt the formation of so-called "shocks". Shock formation, however, cannot arise when the principal part of Eq. (2.1) is semi-linear. It is a special feature of the relativistic gravitational field equations that related FOSH evolution systems *do have*, in the branches that evolve the degrees of freedom in the gravitational field itself, principal parts which are effectively semi-linear.[3] More precisely, the coefficient matrices $M^{AB\,\mu}$ in these branches depend only on those field variables u^A which form a background for gravitational dynamics in the sense of Geroch [15].

B. Choice of gauge source functions and local coordinates

As Friedrich emphasised in Sec. 5.2 of Ref. [11], there exists within a $1+3$ orthonormal frame representation of the relativistic gravitational field equations a set of ten so-called *gauge source functions*, $G := \{\,T^0,\, T^\alpha,\, T^\alpha{}_0,\, T^\alpha{}_\beta\,\}$, that can be *arbitrarily prescribed* in any dynamical consideration (and are thus assumed to be "known"). These relate to (i) the arbitrary choice of a *future-directed* reference "time flow vector field" $\mathbf{T}$,[4] which, in terms of the $1+3$ ONF basis $\{\,\mathbf{e}_0,\,\mathbf{e}_\alpha\,\}$, is expressed by

$$\mathbf{T} := T^0\,\mathbf{e}_0 + T^\alpha\,\mathbf{e}_\alpha \ , \qquad T^0 > 0 \ , \tag{2.6}$$

and (ii) the propagation of the $1+3$ ONF basis $\{\,\mathbf{e}_0,\,\mathbf{e}_\alpha\,\}$ along $\mathbf{T}$, described by

$$\nabla_{\mathbf{T}}\mathbf{e}_0 := T^\alpha{}_0\,\mathbf{e}_\alpha \ , \qquad \nabla_{\mathbf{T}}\mathbf{e}_\alpha := T^0{}_\alpha\,\mathbf{e}_0 + T^\beta{}_\alpha\,\mathbf{e}_\beta \ . \tag{2.7}$$

Parallel transport of $\{\,\mathbf{e}_0,\,\mathbf{e}_\alpha\,\}$ along $\mathbf{T}$, for example, thus corresponds to setting $0 = T^0{}_\alpha = T^\alpha{}_\beta$. Upon introduction of a dimensionless local time coordinate t and dimensionless local spatial coordinates $\{\,x^i\,\}$ that comove with $\mathbf{T}$, the *gauge conditions* related to a $1+3$ orthonormal frame representation are made explicit by [11]

$$e_0{}^\mu = \frac{1}{T^0}\,(M_0^{-1}\,\delta^\mu{}_0 - T^\alpha\,e_\alpha{}^\mu) \ , \qquad \Gamma^0{}_{\alpha 0} = \frac{1}{T^0}\,(T^0{}_\alpha - \Gamma^0{}_{\alpha\beta}\,T^\beta) \ , \qquad \Gamma^\alpha{}_{\beta 0} = \frac{1}{T^0}\,(T^\alpha{}_\beta - \Gamma^\alpha{}_{\beta\gamma}\,T^\gamma) \ ; \tag{2.8}$$

we keep the inverse unit of $[\,\text{length}\,]$, M_0^{-1}, as a coefficient for reasons of physical dimensions.

The fluid-comoving, *Lagrangean* perspective adopted in the discussion of Ref. [9] by identifying the timelike reference congruence with the fluid 4-velocity field, $\mathbf{e}_0 \equiv \mathbf{u}$, is now obtained by fixing three of the four dimensionless coordinate gauge source functions according to $T^\alpha = 0$, resulting in an alignment $\mathbf{T} \parallel \mathbf{e}_0$ ($\equiv \mathbf{u}$). This leads to

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[3] See, e.g., Eqs. (3.21) – (3.23) in Ref. [9], or Eqs. (4.17) – (4.20) in the Abelian G_2 example given in Sec. IV below.

[4] The reference vector field $\mathbf{T}$ need not necessarily be timelike.

$$e_0{}^{\mu} = M^{-1}\,\delta^{\mu}{}_0\ , \qquad T^0{}_{\alpha} = T^0\,\Gamma^0{}_{\alpha 0} = T^0\,\dot{u}_{\alpha}\ , \qquad T^{\alpha}{}_{\beta} = T^0\,\Gamma^{\alpha}{}_{\beta 0} = T^0\,\epsilon^{\alpha}{}_{\beta\gamma}\,\Omega^{\gamma}\ , \tag{2.9}$$

where $M := T^0 M_0$. Consequently, the three frame gauge source functions $T^0{}_{\alpha}$ become proportional to the components of the fluid acceleration $\dot{u}^{\alpha}$, while the three frame gauge source functions $T^{\alpha}{}_{\beta}$ become proportional to the components of the rotation rate Ω^{α} at which the spatial frame $\{\,\mathbf{e}_{\alpha}\,\}$ fails to be Fermi-propagated along $\mathbf{u}$. For cosmological models $(\,\mathcal{M},\,\mathbf{g},\,\mathbf{u}\,)$ with perfect fluid matter sources, Ref. [9] introduced *proper time* along $\mathbf{u}$ by setting $T^0 = 1 \Rightarrow M = M_0$, and derived the evolution equation for $\dot{u}^{\alpha}$ along $\mathbf{u}$ from the commutators on the basis of Eqs. (2.12) and (2.21) below. Additionally, Ref. [9] set $\Omega^{\alpha} = 0$. The latter choice, however, is by no means compulsory, and other choices may prove equally convenient (given that Ω^{α} is assumed to be "known").

In order to obtain from the extended $1+3$ orthonormal frame relations proper partial differential equations such that the theory underlying Subsec. II A applied, Ref. [9] expressed the coordinate components $e_0{}^{\mu} := \mathbf{e}_0(x^{\mu})$ and $e_{\alpha}{}^{\mu} := \mathbf{e}_{\alpha}(x^{\mu})$ of the $1+3$ ONF basis $\{\,\mathbf{e}_0,\,\mathbf{e}_{\alpha}\,\}$ in terms of the comoving local coordinate basis $\{\,\partial_t,\,\partial_i\,\}$ by

$$\mathbf{e}_0 := M^{-1}\,\partial_t\ , \qquad \mathbf{e}_{\alpha} := e_{\alpha}{}^{i}\,(M_i\,\partial_t + \partial_i)\ ; \tag{2.10}$$

$M = M(t, x^i)$ denotes the *threading lapse function* and $M_i\,dx^i = M_i(t, x^j)\,dx^i$ the dimensionless *threading shift 1-form*. The inverse of the *threading metric* is $h^{ij} := \delta^{\alpha\beta}\,e_{\alpha}{}^{i}\,e_{\beta}{}^{j}$. See, e.g., Ref. [10] and references therein.

C. Matter model

The matter sources in the present discussion (as well as in Ref. [9]) are assumed to be modelled phenomenologically as a *perfect fluid* such that, with respect to fluid-comoving observers,

$$0 = q^{\alpha}(\mathbf{u}) = \pi_{\alpha\beta}(\mathbf{u})\ , \tag{2.11}$$

i.e., the energy current density and the anisotropic pressure both vanish. Additionally, a *baryotropic* equation of state is assumed,

$$p = p(\mu)\ , \tag{2.12}$$

relating the isotropic pressure $p(\mathbf{u})$ to the total energy density $\mu(\mathbf{u})$.

D. Constraint equations

The following relations in the set obtained from an extended 1+3 orthonormal frame representation of the relativistic gravitational field equations do *not* contain any frame derivatives with respect to $\mathbf{e}_0$. Hence, it is commonplace to refer to these relations as "constraint equations".[5] These are [19,10]

$$0 = (C_1)^{\alpha} \ := \ (\mathbf{e}_{\beta} - 3\,a_{\beta})\,(\sigma^{\alpha\beta}) - \tfrac{2}{3}\,\delta^{\alpha\beta}\,\mathbf{e}_{\beta}(\Theta) - n^{\alpha}{}_{\beta}\,\omega^{\beta} + \epsilon^{\alpha\beta\gamma}\,[\,(\mathbf{e}_{\beta} + 2\,\dot{u}_{\beta} - a_{\beta})\,(\omega_{\gamma}) - n_{\beta\delta}\,\sigma^{\delta}{}_{\gamma}\,] \tag{2.13}$$

$$0 = (C_2) \ := \ (\mathbf{e}_{\alpha} - \dot{u}_{\alpha} - 2\,a_{\alpha})\,(\omega^{\alpha}) \tag{2.14}$$

$$\begin{aligned} 0 = (C_3)^{\alpha\beta} \ := \ & (\delta^{\gamma\langle\alpha}\,\mathbf{e}_{\gamma} + 2\,\dot{u}^{\langle\alpha} + a^{\langle\alpha})\,(\omega^{\beta\rangle}) - \tfrac{1}{2}\,n^{\gamma}{}_{\gamma}\,\sigma^{\alpha\beta} + 3\,n^{\langle\alpha}{}_{\gamma}\,\sigma^{\beta\rangle\gamma} + H^{\alpha\beta} \\ & - \epsilon^{\gamma\delta\langle\alpha}\,[\,(\mathbf{e}_{\gamma} - a_{\gamma})\,(\sigma^{\beta\rangle}{}_{\delta}) + n^{\beta\rangle}{}_{\gamma}\,\omega_{\delta}\,] \end{aligned} \tag{2.15}$$

$$0 = (C_{\mathrm{J}})^{\alpha} \ := \ (\mathbf{e}_{\beta} - 2\,a_{\beta})\,(n^{\alpha\beta}) + \tfrac{2}{3}\,\Theta\,\omega^{\alpha} + 2\,\sigma^{\alpha}{}_{\beta}\,\omega^{\beta} + \epsilon^{\alpha\beta\gamma}\,[\,\mathbf{e}_{\beta}(a_{\gamma}) - 2\,\omega_{\beta}\,\Omega_{\gamma}\,] \tag{2.16}$$

$$0 = (C_{\mathrm{G}})^{\alpha\beta} \ := \ {}^{*}\!S^{\alpha\beta} + \tfrac{1}{3}\,\Theta\,\sigma^{\alpha\beta} - \sigma^{\langle\alpha}{}_{\gamma}\,\sigma^{\beta\rangle\gamma} - \omega^{\langle\alpha}\,\omega^{\beta\rangle} + 2\,\omega^{\langle\alpha}\,\Omega^{\beta\rangle} - E^{\alpha\beta} \tag{2.17}$$

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[5] Even though this terminology is problematic in the generic case when $\mathbf{e}_0 \equiv \mathbf{u}$ has *non-zero* vorticity, $\omega^{\alpha}(\mathbf{u}) \neq 0$, and local coordinates are introduced according to Eq. (2.10) above.

$$0 = (C_{\rm G}) \;:=\; {}^*R + \tfrac{2}{3}\,\Theta^2 - (\sigma_{\alpha\beta}\sigma^{\alpha\beta}) + 2\,(\omega_\alpha\omega^\alpha) - 4\,(\omega_\alpha\,\Omega^\alpha) - 2\,\mu - 2\,\Lambda \tag{2.18}$$

$$0 = (C_4)^\alpha \;:=\; (\mathbf{e}_\beta - 3\,a_\beta)\,(E^{\alpha\beta}) - \tfrac{1}{3}\,\delta^{\alpha\beta}\,\mathbf{e}_\beta(\mu) - 3\,\omega_\beta\,H^{\alpha\beta} - \epsilon^{\alpha\beta\gamma}\,[\;\sigma_{\beta\delta}\,H^\delta{}_\gamma + n_{\beta\delta}\,E^\delta{}_\gamma\;] \tag{2.19}$$

$$0 = (C_5)^\alpha \;:=\; (\mathbf{e}_\beta - 3\,a_\beta)\,(H^{\alpha\beta}) + (\mu+p)\,\omega^\alpha + 3\,\omega_\beta\,E^{\alpha\beta} + \epsilon^{\alpha\beta\gamma}\,[\;\sigma_{\beta\delta}\,E^\delta{}_\gamma - n_{\beta\delta}\,H^\delta{}_\gamma\;] \tag{2.20}$$

$$0 = (C_{\rm PF})^\alpha \;:=\; c_s^2\,\delta^{\alpha\beta}\,\mathbf{e}_\beta(\mu) + (\mu+p)\,\dot{u}^\alpha \;, \tag{2.21}$$

where

$${}^*S_{\alpha\beta} := \mathbf{e}_{\langle\alpha}(a_{\beta\rangle}) + b_{\langle\alpha\beta\rangle} - \epsilon^{\gamma\delta}{}_{\langle\alpha}\,(\mathbf{e}_{|\gamma|} - 2\,a_{|\gamma|})\,(n_{\beta\rangle\delta}) \tag{2.22}$$

$${}^*R := 2\,(2\,\mathbf{e}_\alpha - 3\,a_\alpha)\,(a^\alpha) - \tfrac{1}{2}\,b^\alpha{}_\alpha \tag{2.23}$$

$$b_{\alpha\beta} := 2\,n_{\alpha\gamma}\,n_\beta{}^\gamma - n^\gamma{}_\gamma\,n_{\alpha\beta} \;, \tag{2.24}$$

$c_s^2(\mu) := dp(\mu)/d\mu$ defines the *isentropic speed of sound* with $0 \leq c_s^2 \leq 1$, and angle brackets denote the symmetric tracefree part.[6] While Eqs. (2.13) – (2.18) derive from the Ricci identities, the Jacobi identities and the Einstein field equations and so are of order $\partial^2 g$, Eqs. (2.19) – (2.21) derive from the once- and twice-contracted second Bianchi identities and so are of order $\partial^3 g$. The divergence constraint equation (2.13) for the fluid rate of shear is often referred to as the "momentum constraint". When $0 = \omega^\alpha(\mathbf{u})$, such that the fluid 4-velocity field $\mathbf{u}$ constitutes the normals to a family of spacelike 3-surfaces $\mathcal{S}$:$\{t = \text{const}\}$, Eqs. (2.17) and (2.18) correspond to the symmetric tracefree and trace parts of the once-contracted Gauß embedding equation. In this case, one also speaks of (C_G) as the generalised Friedmann equation, alias the "Hamiltonian constraint" or the "energy constraint".

E. $(1+1+2)$–decomposition

In the present discussion it proves very helpful to consider a $(1+1+2)$–decomposition of all geometrically defined field variables and their dynamical relations. In order to do so, we arbitrarily pick the frame basis field $\mathbf{e}_1$ as a second, spacelike, reference direction, in addition to $\mathbf{e}_0 \equiv \mathbf{u}$ as a timelike one; any other spatial direction, however, would be equally acceptable. Hence, in a small isotropic neighbourhood $\mathcal{U}$ in the local rest 3-space of an arbitrary event $\mathcal{P}$, we establish the convention of regarding those spatial frame components of geometrical objects which contain the index "1" as (semi-)*longitudinal* with respect to $\mathbf{e}_1$, while regarding those which exclude the index "1" as *transverse* with respect to $\mathbf{e}_1$. Likewise, in $\mathcal{U}$, $\mathbf{e}_1$ shall constitute the *outward* frame derivative while $\mathbf{e}_2$ and $\mathbf{e}_3$ shall be *tangential* frame derivatives when we turn in Sec. III to the issue of jump discontinuities across spherical spacelike 2-surfaces $\mathcal{J}$:$\{t = \text{const}, \phi(x^\mu) = \text{const}\}$. For the frame components of spatial rank-2 symmetric tracefree tensors $a_{\alpha\beta} = a_{\langle\alpha\beta\rangle}$ with squared magnitude $a^2 := \frac{1}{2}(a_{\alpha\beta}a^{\alpha\beta}) \geq 0$, we define a *new* set of frame variables by

$$\begin{array}{lll} a_+ := \tfrac{1}{2}\,(a_{22} + a_{33}) = -\tfrac{1}{2}\,a_{11} & a_- := \tfrac{1}{2\sqrt{3}}\,(a_{22} - a_{33}) & \\ a_\times := \tfrac{1}{\sqrt{3}}\,a_{23} & a_2 := \tfrac{1}{\sqrt{3}}\,a_{31} & a_3 := \tfrac{1}{\sqrt{3}}\,a_{12} \;, \end{array} \tag{2.25}$$

so that

$$a^2 = 3\,(a_+^2 + a_-^2 + a_\times^2 + a_2^2 + a_3^2) \;. \tag{2.26}$$

In particular, in the present discussion we have $a_{\alpha\beta} \in \{\,\sigma_{\alpha\beta},\, E_{\alpha\beta},\, H_{\alpha\beta}\,\}$. We remark that these definitions are now adapted to the conventions of the book edited by Wainwright and Ellis [32], implying they differ by a factor of $\frac{1}{3}$ from those used in Ref. [9]. In analogy to Eq. (2.25), we perform a $(1+1+2)$–decomposition of the spatial commutation functions $n_{\alpha\beta}$ by defining

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[6] The numbering of the constraint equations we employ is based on the conventions established in $1+3$ covariant treatments of relativistic cosmological models $(\,\mathcal{M},\, \mathbf{g},\, \mathbf{u}\,)$ (cf. Ref. [7]).

$$
\begin{aligned}
&n := n_{11} + n_{22} + n_{33} \quad && n_{+} := -\,n_{11} + \tfrac{1}{2}\,(n_{22} + n_{33}) \quad && n_{-} := \tfrac{1}{2\sqrt{3}}\,(n_{22} - n_{33}) \\
&n_{\times} := \tfrac{1}{\sqrt{3}}\,n_{23} && n_{2} := \tfrac{1}{\sqrt{3}}\,n_{31} && n_{3} := \tfrac{1}{\sqrt{3}}\,n_{12}\ .
\end{aligned}
\tag{2.27}
$$

The squared magnitude is then given by

$$
\tfrac{1}{2}(n_{\alpha\beta}n^{\alpha\beta}) = \tfrac{1}{6}\,(n^2 + 2n_{+}^2) + 3\,(n_{-}^2 + n_{\times}^2 + n_{2}^2 + n_{3}^2)\ . \tag{2.28}
$$

Note that only $(n - 2n_{+})$, n_{-} and $n_{\times}$ transform as *tensor components* under rotations of the spatial frame $\{\,\mathbf{e}_\alpha\,\}$ about the reference $\mathbf{e}_1$-axis.

By employing these conventions and definitions, we have listed in Appendix A 2 certain linear combinations of the components of the constraint equations (2.13) – (2.21) that will be needed in the following sections.

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INSERT Table I on "Conventions for $(1+1+2)$–decomposition" HERE.

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III. PHYSICAL EFFECT OF CONSTRAINT EQUATIONS ON OUTWARD FIRST DERIVATIVES

Generic cosmological models $(\,\mathcal{M},\,\mathbf{g},\,\mathbf{u}\,)$ with a perfect fluid matter source have fluid 4-velocity fields $\mathbf{u}$ with *non-zero* vorticity, $\omega^\alpha(\mathbf{u}) \neq 0$. This property makes it impossible to determine a fluid-comoving spacelike 3-surface $\mathcal{S}$:$\{t = \text{const}\}$ *everywhere* orthogonal to $\mathbf{u}$ on which initial data satisfying the constraint equations of Subsec. II D could be specified. In this case, the discussion of a well-posed Cauchy initial value problem requires that the setting of the data as well as the solution of the constraint equations be instead performed on a *non-comoving* spacelike 3-surface $\mathcal{S}$:$\{t = \text{const}\}$, before the data is evolved along $\mathbf{u}$.[7] All these complications disappear when $0 = \omega^\alpha(\mathbf{u})$, and well-defined spacelike 3-surface $\mathcal{S}$:$\{t = \text{const}\}$ everywhere orthogonal to $\mathbf{u}$ *do* exist.

For our purposes, however, it is sufficient to investigate the physical effect of the constraint equations on the selection of appropriate initial data sets from a purely *local* viewpoint, i.e., only in a small isotropic neighbourhood $\mathcal{U}$ in the local rest 3-space of an arbitrary event $\mathcal{P}$. Due to the local Minkowskian structure of all relativistic spacetime manifolds $(\,\mathcal{M},\,\mathbf{g}\,)$, one conventionally determines (and analyses their physical properties) the set of characteristic cones $\mathcal{C}$:$\{\phi(x^\mu) = \text{const}\}$ for a given FOSH evolution system only within the small isotropic spacetime neighbourhood $\{-\,\varepsilon \leq t \leq \varepsilon\} \times \mathcal{U}$ of $\mathcal{P}$. In line with this, we will consider in the following spherical spacelike 2-surfaces $\mathcal{J}$:$\{t = \text{const}, \phi(x^\mu) = \text{const}\}$ in $\mathcal{U}$ across which we assume to exist (i) *jump discontinuities* in the *outward first frame derivatives* of certain geometrically defined field variables u^A, and (ii) *continuity* of the tangential first frame derivatives of the u^A and the u^A themselves. As the constraint equations have to be satisfied *everywhere*, it is clear that across $\mathcal{J}$ we have $0 = [(C)^{\alpha\cdots}]_{t,\phi=\text{const}}$ for the value of any component in the set of Eqs. (2.13) – (2.21).

Motivated by the prospect of grounding the discussion on wave-like phenomena described by the relativistic gravitational field equations on the deviation equation for a set of test particles, the dynamical considerations on the $\partial^3 g$-order FOSH evolution system in Ref. [9] focused on the set of Weyl curvature characteristic eigenfields $\{\,E_{+},\,H_{+},\,(E_3 \mp H_2),\,(E_2 \pm H_3),\,(E_{-} \mp H_{\times}),\,(E_{\times} \pm H_{-})\,\}$. There it was correctly argued that the deviation equation monitors the physical effects on the state of motion of a set of test particles of both *gradual* as well as *sudden* changes in the values of these fields (see, e.g., Refs. [22] and [26]). What was overlooked in this work, however, is the fact that, on the basis of the theory underlying Subsec. II A, the $\partial^3 g$-order FOSH evolution system presented in Ref. [9] can at best describe the physical transport along bicharacteristic rays of jump discontinuities in the (outward) first *derivatives* of these fields rather than these fields themselves, given the constraint equations do not impose any additional restrictions. It is our aim to supplement the discussion of Ref. [9] by such a consideration in the present section.

A. Jump discontinuities at derivative level $\partial^3 g$

Considering in $\mathcal{U}$ a spherical spacelike 2-surface $\mathcal{J}$:$\{t = \text{const}, \phi(x^\mu) = \text{const}\}$, and assuming that across $\mathcal{J}$ all geometrically defined field variables u^A as well as their tangential first frame derivatives are continuous, we find that

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[7] It is currently unknown whether the $\partial^3 g$-order FOSH evolution systems with perfect fluid matter sources in Refs. [12] and [9] can be generalised to a non-comoving perspective.

the Weyl curvature divergence equations (A11) – (A14) amongst the constraint equations lead to the following set of jump conditions:

$$\begin{aligned}
[\mathbf{e}_1(E_+)]_{t,\phi=\mathrm{const}} &= -\tfrac{1}{6}\,[\mathbf{e}_1(\mu)]_{t,\phi=\mathrm{const}} &(3.1)\\
[\mathbf{e}_1(H_+)]_{t,\phi=\mathrm{const}} &= 0 &(3.2)\\
[\mathbf{e}_1(E_3 \mp H_2)]_{t,\phi=\mathrm{const}} &= 0 &(3.3)\\
[\mathbf{e}_1(E_2 \pm H_3)]_{t,\phi=\mathrm{const}} &= 0 &(3.4)\\
[\mathbf{e}_1(E_- \mp H_\times)]_{t,\phi=\mathrm{const}} &= \text{unconstrained} &(3.5)\\
[\mathbf{e}_1(E_\times \pm H_-)]_{t,\phi=\mathrm{const}} &= \text{unconstrained}\ . &(3.6)
\end{aligned}$$

The implications are three-fold:

(i) Jump discontinuities in the outward first frame derivative of the Coulomb-like Weyl curvature characteristic eigenfield E_+ originate from jump discontinuities in the outward first frame derivative of the matter total energy density μ and are physically allowed, if the momentum conservation equation (A15) is satisfied on both sides of $\mathcal{J}$ with $0 = [(C_{\mathrm{PF}})_1]_{t,\phi=\mathrm{const}}$. In more detail: assuming that all of p, c_s and $\dot{u}_1$ are continuous across $\mathcal{J}$, Eq. (A15) yields

$$0 = [(C_{\mathrm{PF}})_1]_{t,\phi=\mathrm{const}} = c_s^2\ [\mathbf{e}_1(\mu)]_{t,\phi=\mathrm{const}} + [\mu]_{t,\phi=\mathrm{const}}\ \dot{u}_1\ . \tag{3.7}$$

Hence, if $p = 0 \Rightarrow c_s = 0$ and $\dot{u}_1 = 0$ on both sides of $\mathcal{J}$, the values of $[\mathbf{e}_1(\mu)]_{t,\phi=\mathrm{const}}$ and $[\mu]_{t,\phi=\mathrm{const}}$ remain unconstrained. Phenomena of the presently described kind occur, for example, across the surfaces of static, spherically symmetrical perfect fluid stars with equation of state (2.12) (see, e.g., Ref. [21]). In a $\partial^3 g$-order formulation, it follows that real-valued initial data for μ (and so for E_+) is required to be of differentiability class $C^2(\mathcal{U})$ with respect to the zeroth-order derivative level of $\mathbf{g}$. The equations of Ref. [9] show that $[\mathbf{e}_1(E_+)]_{t,\phi=\mathrm{const}} \neq 0$ propagates with characteristic velocity $v = 0$ relative to $\mathbf{u}$. Note that in the vacuum subcase $\mu = 0 \Rightarrow [\mathbf{e}_1(E_+)]_{t,\phi=\mathrm{const}} = 0$. Jump discontinuities in $\mathbf{e}_1(H_+)$ are not physically allowed, and so, in a $\partial^3 g$-order formulation, real-valued initial data for H_+ is required to be of differentiability class $C^3(\mathcal{U})$ with respect to the zeroth-order derivative level of $\mathbf{g}$.

(ii) Jump discontinuities in the outward first frame derivatives of the semi-longitudinal Weyl curvature characteristic eigenfields $(E_3 \mp H_2)$ and $(E_2 \pm H_3)$, that by restricting to the net $\partial^3 g$-order FOSH evolution system in Ref. [9] (*without* accounting for the constraint equations) are theoretically associated with characteristic velocities $v = \pm\frac{1}{2}$ relative to $\mathbf{u}$, are *not physically allowed*. Hence, in a $\partial^3 g$-order formulation, real-valued initial data for $(E_3 \mp H_2)$ and $(E_2 \pm H_3)$ needs to be of differentiability class $C^3(\mathcal{U})$ (rather than $C^2(\mathcal{U})$) with respect to the zeroth-order derivative level of $\mathbf{g}$. In short, *not all* initial data can be given freely.

It can be easily inferred from the propagation equations along $\mathbf{u}$ for the Weyl curvature divergence equations (2.19) and (2.20), first published for an irrotational pressure-free fluid matter source in Ref. [18], and presented for a general perfect fluid matter source in Refs. [8] and [13], that the characteristic velocities relative to $\mathbf{u}$ for the components $(C_4)_2 \mp (C_5)_3$ and $(C_4)_3 \pm (C_5)_2$ are $v = \pm\frac{1}{2}$ too. Hence, comparing this result with Eqs. (A13) and (A14) in Appendix A 2 and Eqs. (3.3) and (3.4) above, it becomes clear that the Weyl curvature divergence equations propagate relative to $\mathbf{u}$ at *precisely* the speed that is required to ensure that jump discontinuities in $\mathbf{e}_1(E_3 \mp H_2)$ and $\mathbf{e}_1(E_2 \pm H_3)$ will *always remain physically disallowed* at any instant throughout the dynamical evolution of a cosmological model $(\mathcal{M}, \mathbf{g}, \mathbf{u})$. It should be emphasised at this stage that this property is completely independent of the presence of matter. That is, of course the jump conditions (3.3) and (3.4) apply equally to vacuum spacetime configurations.

(iii) Jump discontinuities in the outward first frame derivatives of the transverse Weyl curvature characteristic eigenfields $(E_- \mp H_\times)$ and $(E_\times \pm H_-)$ are physically allowed. Clearly, this situation reflects the freedom of specifying four *arbitrary* (non-analytic) real-valued functions $I_{\partial^3 g} := \{\, a_1(x^i),\, a_2(x^i),\, a_3(x^i),\, a_4(x^i)\,\}$ of differentiability class $C^2(\mathcal{U})$ with respect to the zeroth-order derivative level of $\mathbf{g}$ as the initial data for the dynamical degrees of freedom associated with the gravitational field itself.

B. Jump discontinuities at derivative level $\partial^2 g$

To be able to argue in terms of physical effects described by the deviation equation for a set of test particles, we have to turn our attention directly to the set of Weyl curvature characteristic eigenfields and possible discontinuous changes in their values. Such changes are driven by the dynamics of the underlying connection fields and their

derivatives. Hence, to facilitate the interpretation of generic gravitational dynamics, it would be desirable to have available a $\partial^2 g$-order FOSH evolution system derived from a $1+3$ orthonormal frame *connection* formulation of gravitational fields (see Refs. [19] and [10] for reviews of the latter). Unfortunately, to date, such a formulation has not been accomplished for the generic case. The exception is the $\partial^2 g$-order FOSH evolution system for perfect fluid cosmological models $(\,\mathcal{M},\,\mathbf{g},\,\mathbf{u}\,)$ with an Abelian G_2 isometry group we will present in Sec. IV below. In the absence of such a generally applicable dynamical formulation, we return to our local viewpoint and investigate how, in a small isotropic neighbourhood $\mathcal{U}$ in the local rest 3-space of an arbitrary event $\mathcal{P}$, the constraint equations at derivative level $\partial^2 g$ restrict the occurrence of jump discontinuities in the values of the Weyl curvature characteristic eigenfields themselves. To this end, we now focus on Eqs. (2.13) – (2.18), and certain linear combinations of the components thereof provided by Eqs. (A5) – (A10) and (A18) – (A23). Again, we consider in $\mathcal{U}$ a spherical spacelike 2-surface $\mathcal{J}$:$\{t = \text{const}, \phi(x^\mu) = \text{const}\}$, and assume that across $\mathcal{J}$ all geometrically defined field variables u^A as well as their tangential first frame derivatives are continuous. This leads to:

(i) $v = 0$ *longitudinal Weyl curvature characteristic eigenfields*:

$$E_+ = -\tfrac{1}{3}\,\mathbf{e}_1(a_1) + \text{tangential frame derivatives/algebraic terms} \tag{3.8}$$
$$H_+ = \tfrac{1}{3}\,\mathbf{e}_1(\omega_1) + \text{tangential frame derivatives/algebraic terms}\ , \tag{3.9}$$

from Eqs. (A18) and (A19). Across $\mathcal{J}$, the generalised Gauß–Friedmann equation (A10) and the fluid vorticity divergence equation (A9), respectively, then impose the restrictions

$$[E_+]_{t,\phi=\text{const}} = -\tfrac{1}{3}\,[\mathbf{e}_1(a_1)]_{t,\phi=\text{const}} = -\tfrac{1}{6}\,[\mu]_{t,\phi=\text{const}} \tag{3.10}$$
$$[H_+]_{t,\phi=\text{const}} = \tfrac{1}{3}\,[\mathbf{e}_1(\omega_1)]_{t,\phi=\text{const}} = 0\ . \tag{3.11}$$

That is, jump discontinuities in the values of the Coulomb-like Weyl curvature characteristic eigenfield E_+ originate from jump discontinuities in the values of the matter total energy density μ and are physically allowed, if the momentum conservation equation (A15) is satisfied on both sides of $\mathcal{J}$ with $0 = [(C_{\rm PF})_1]_{t,\phi=\text{const}}$ [cf. Eq. (3.7)]. In a $\partial^2 g$-order formulation, real-valued initial data for μ (and so for E_+) is thus required to be of differentiability class $C^1(\mathcal{U})$ with respect to the zeroth-order derivative level of $\mathbf{g}$. Note that in the vacuum subcase $\mu = 0 \Rightarrow [E_+]_{t,\phi=\text{const}} = 0$. Jump discontinuities in the values of H_+ are not physically allowed, and so, in a $\partial^2 g$-order formulation, real-valued initial data for H_+ is required to be of differentiability class $C^2(\mathcal{U})$ with respect to the zeroth-order derivative level of $\mathbf{g}$.

(ii) $v = \pm\,\frac{1}{2}$ *semi-longitudinal Weyl curvature characteristic eigenfields*:

$$(E_3 \mp H_2) = \mp\,\tfrac{1}{2}\,\mathbf{e}_1(\sigma_3 \mp n_2 - \tfrac{1}{\sqrt{3}}\,\omega_3 \mp \tfrac{1}{\sqrt{3}}\,a_2) + \text{tangential frame derivatives/algebraic terms} \tag{3.12}$$
$$(E_2 \pm H_3) = \mp\,\tfrac{1}{2}\,\mathbf{e}_1(\sigma_2 \pm n_3 + \tfrac{1}{\sqrt{3}}\,\omega_2 \mp \tfrac{1}{\sqrt{3}}\,a_3) + \text{tangential frame derivatives/algebraic terms}\ , \tag{3.13}$$

from Eqs. (A20) and (A21). In this case, we find that across $\mathcal{J}$ the fluid shear divergence/Jacobi constraint equations (A7) and (A8) impose the restrictions

$$[(E_3 \mp H_2)]_{t,\phi=\text{const}} = \mp\,\tfrac{1}{2}\,[\mathbf{e}_1(\sigma_3 \mp n_2 - \tfrac{1}{\sqrt{3}}\,\omega_3 \mp \tfrac{1}{\sqrt{3}}\,a_2)]_{t,\phi=\text{const}} = 0 \tag{3.14}$$
$$[(E_2 \pm H_3)]_{t,\phi=\text{const}} = \mp\,\tfrac{1}{2}\,[\mathbf{e}_1(\sigma_2 \pm n_3 + \tfrac{1}{\sqrt{3}}\,\omega_2 \mp \tfrac{1}{\sqrt{3}}\,a_3)]_{t,\phi=\text{const}} = 0\ . \tag{3.15}$$

That is, jump discontinuities in the values of the semi-longitudinal Weyl curvature characteristic eigenfields $(E_3 \mp H_2)$ and $(E_2 \pm H_3)$ are *not physically allowed*. In a $\partial^2 g$-order formulation real-valued initial data for $(E_3 \mp H_2)$ and $(E_2 \pm H_3)$ is thus required to be of differentiability class $C^2(\mathcal{U})$ (rather than $C^1(\mathcal{U})$) with respect to the zeroth-order derivative level of $\mathbf{g}$.

(iii) $v = \pm\,1$ *transverse Weyl curvature characteristic eigenfields*:

$$(E_- \mp H_\times) = \mp\,\mathbf{e}_1(\sigma_- \mp n_\times) + \text{tangential frame derivatives/algebraic terms} \tag{3.16}$$
$$(E_\times \pm H_-) = \mp\,\mathbf{e}_1(\sigma_\times \pm n_-) + \text{tangential frame derivatives/algebraic terms}\ , \tag{3.17}$$

from Eqs. (A22) and (A23). In this case, we find that across $\mathcal{J}$ the constraint equations impose *no* restrictions so that

$$[(E_- \mp H_\times)]_{t,\phi=\text{const}} = \mp\,[\mathbf{e}_1(\sigma_- \mp n_\times)]_{t,\phi=\text{const}} = \text{unconstrained} \tag{3.18}$$
$$[(E_\times \pm H_-)]_{t,\phi=\text{const}} = \mp\,[\mathbf{e}_1(\sigma_\times \pm n_-)]_{t,\phi=\text{const}} = \text{unconstrained}\ . \tag{3.19}$$

That is, jump discontinuities in the values of the transverse Weyl curvature characteristic eigenfields $(E_{-} \mp H_{\times})$ and $(E_{\times} \pm H_{-})$ are physically allowed. Hence, they can transport arbitrary non-zero jump discontinuities $[\mathbf{e}_1(\sigma_{-} \mp n_{\times})]_{t,\phi=\text{const}}$ and $[\mathbf{e}_1(\sigma_{\times} \pm n_{-})]_{t,\phi=\text{const}}$ of finite magnitude. Again, this reflects the freedom of specifying four *arbitrary* (non-analytic) real-valued functions $I_{\partial^2 g} := \{\, A_1(x^i),\, A_2(x^i),\, A_3(x^i),\, A_4(x^i)\,\}$ of differentiability class $C^1(\mathcal{U})$ with respect to the zeroth-order derivative level of $\mathbf{g}$ as the initial data for the dynamical degrees of freedom associated with the gravitational field itself.

For completeness, we now also briefly discuss the effect of the constraint equations (2.13) – (2.16), respectively, Eqs. (A5) – (A9), on the outward first frame derivatives of the characteristic eigenfields associated with the fluid kinematical branch of the $\partial^3 g$-order FOSH evolution system in Ref. [9] [cf. Eqs. (3.27) and (3.28)]. The derivative level is hence one below the Weyl curvature case. For the $v = 0$ fluid kinematical characteristic eigenfields we find the jump conditions:

(iv) $v = 0$ *fluid kinematical characteristic eigenfields*:

$$[\mathbf{e}_1(\tfrac{1}{3}\,\Theta + \sigma_{+})]_{t,\phi=\text{const}} = 0 \tag{3.20}$$

$$[\mathbf{e}_1(\omega_1)]_{t,\phi=\text{const}} = 0 \tag{3.21}$$

$$[\mathbf{e}_1(\sigma_3 + \tfrac{1}{\sqrt{3}}\,\omega_3)]_{t,\phi=\text{const}} = \text{unconstrained} \tag{3.22}$$

$$[\mathbf{e}_1(\sigma_2 - \tfrac{1}{\sqrt{3}}\,\omega_2)]_{t,\phi=\text{const}} = \text{unconstrained}\ . \tag{3.23}$$

The last two conditions imply the existence of two generically non-zero *fluid rotational modes* that were identified before by Ehlers et al in an analysis of linearised perturbations of arbitrary background dust spacetimes [4]. Corresponding real-valued initial data for these modes is required to be of differentiability class $C^1(\mathcal{U})$ with respect to the zeroth-order derivative level of $\mathbf{g}$. Finally, for the different parts of the $v = \pm\, c_s$ fluid kinematical characteristic eigenfields we find the jump conditions:

(v) $v = \pm\, c_s$ *fluid kinematical characteristic eigenfields*:

$$[\mathbf{e}_1(\tfrac{1}{3}\,\Theta - 2\,\sigma_{+})]_{t,\phi=\text{const}} = \text{unconstrained} \tag{3.24}$$

$$[\mathbf{e}_1(\sigma_3 - \tfrac{1}{\sqrt{3}}\,\omega_3)]_{t,\phi=\text{const}} = \pm\,[\mathbf{e}_1(\tfrac{1}{\sqrt{3}}\,a_2 + n_2)]_{t,\phi=\text{const}} = \text{unconstrained} \tag{3.25}$$

$$[\mathbf{e}_1(\sigma_2 + \tfrac{1}{\sqrt{3}}\,\omega_2)]_{t,\phi=\text{const}} = \pm\,[\mathbf{e}_1(\tfrac{1}{\sqrt{3}}\,a_3 - n_3)]_{t,\phi=\text{const}} = \text{unconstrained} \tag{3.26}$$

$$[\mathbf{e}_1(\dot{u}_1)]_{t,\phi=\text{const}} = \text{unconstrained} \tag{3.27}$$

$$[\mathbf{e}_1(\dot{u}_2)]_{t,\phi=\text{const}} = \text{unconstrained} \tag{3.28}$$

$$[\mathbf{e}_1(\dot{u}_3)]_{t,\phi=\text{const}} = \text{unconstrained}\ . \tag{3.29}$$

Again, corresponding real-valued initial data for these parts is required to be of differentiability class $C^1(\mathcal{U})$ with respect to the zeroth-order derivative level of $\mathbf{g}$. Note especially that the jump conditions (3.25) and (3.26) are precisely of such a nature that *no* violations of the jump conditions (3.14) and (3.15) above for the semi-longitudinal Weyl curvature characteristic eigenfields may occur.

IV. WORKED EXAMPLE: COSMOLOGICAL MODELS WITH ABELIAN G_2 ISOMETRY GROUP

In this section we turn to discuss in some detail a *new*, fully gauge-fixed, $\partial^2 g$-order autonomous FOSH evolution system for spatially inhomogeneous perfect fluid cosmological models $(\,\mathcal{M},\, \mathbf{g},\, \mathbf{u}\,)$ which are invariant under the transformations of an *Abelian* G_2 *isometry group* that is simply transitive on spacelike 2-surfaces. Thus, all geometrically defined field variables u^A vary in *one* spatial direction only. A systematic approach to this class of cosmological models was brought forward some time ago by Wainwright in Refs. [30] and [31], wherein the generic case was given the classification label "A(i)". A number of exact, real analytic, solutions to the EFE for Abelian G_2 perfect fluid cosmological models are known, such as those listed in Refs. [31] and [32] or the singularity-free solution obtained by Senovilla [25], but most of them belong to dynamically restricted or higher-symmetry subcases.

A. Well-posed Cauchy initial value problem

Choosing an *orbit-aligned group-invariant* $1 + 3$ ONF basis $\{\, \mathbf{e}_0,\, \mathbf{e}_\alpha\,\}$ such that commutation relations

$$0 = [\,\boldsymbol{\xi}, \mathbf{e}_0\,] = [\,\boldsymbol{\xi}, \mathbf{e}_\alpha\,] = [\,\boldsymbol{\eta}, \mathbf{e}_0\,] = [\,\boldsymbol{\eta}, \mathbf{e}_\alpha\,] \tag{4.1}$$

hold between the two commuting spacelike *Killing vector fields* $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ and $\{\,\mathbf{e}_0,\,\mathbf{e}_\alpha\,\}$, and assuming that $\mathbf{e}_0 \equiv \mathbf{u}$ is *orthogonal* to the isometry group orbits, it follows that for all solutions in the Abelian G_2 class we have [30,31]

$$0 = \mathbf{e}_2(u^A) = \mathbf{e}_3(u^A)\ , \qquad 0 = \dot{u}_2 = \dot{u}_3 = \omega^\alpha(\mathbf{u}) = a_2 = a_3 = (n - 2n_+) = n_2 = n_3\ . \tag{4.2}$$

That is, $\mathbf{e}_2$ and $\mathbf{e}_3$ are tangent to the isometry group orbits. Besides $\mathbf{e}_0 \equiv \mathbf{u}$ also the frame basis field $\mathbf{e}_1$ is *hypersurface orthogonal* [30]. The *canonical choice* Wainwright proposes for Abelian G_2 perfect fluid cosmological models consists of introducing fluid-comoving local coordinates $\{\,t,\,x,\,y,\,z\,\}$ adapted to the integral curves of $\mathbf{u}$, $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ such that [31]

$$\boldsymbol{\xi} = \partial_y\ , \qquad \boldsymbol{\eta} = \partial_z\ ; \tag{4.3}$$

the isometry group orbits, which have vanishing Gaußian 2-curvature, are thus given by spacelike 2-surfaces $\{t = \text{const}, x = \text{const}\}$. Additionally, the coordinate components of the $1+3$ ONF basis $\{\,\mathbf{e}_0,\,\mathbf{e}_\alpha\,\}$ as introduced by Eq. (2.10) are specialised to [31]

$$\begin{aligned} \mathbf{e}_0 &= M^{-1}\,\partial_t\ , & \mathbf{e}_2 &= {e_2}^2\,\partial_y\ , \\ \mathbf{e}_1 &= {e_1}^1\,\partial_x + {e_1}^2\,\partial_y + {e_1}^3\,\partial_z\ , & \mathbf{e}_3 &= {e_3}^2\,\partial_y + {e_3}^3\,\partial_z\ ; \end{aligned} \tag{4.4}$$

in view of $0 = \omega^\alpha(\mathbf{u})$ the choice $0 = M_i$ is made.[8] Note that Wainwright's canonical choice establishes the property $\mathbf{e}_2 \parallel \boldsymbol{\xi}$. All geometrically defined field variables u^A will now only be functions of the local coordinates t and x. From the commutators (see, e.g., Refs. [19] and [10]), the canonical choice of $1+3$ ONF basis and local coordinates has the direct consequences

$$0 = (\sqrt{3}\,\sigma_\times + \Omega_1) = (\sqrt{3}\,\sigma_2 + \Omega_2) = (\sqrt{3}\,\sigma_3 - \Omega_3)\ , \tag{4.5}$$

implying the three spatial coordinate conditions $0 = {e_2}^1 = {e_2}^3 = {e_3}^1$ in Eq. (4.4) will be automatically preserved along $\mathbf{u}$. Thus, the spatial frame $\{\,\mathbf{e}_\alpha\,\}$ will presently *not* be Fermi-transported along $\mathbf{u}$, but Ω^α (and so the three frame gauge source functions ${T^\alpha}_\beta$) will be adapted to the fluid rate of shear instead. Likewise, the canonical choice leads to

$$0 = (n_+ - \sqrt{3}\,n_-)\ . \tag{4.6}$$

We now define two new frame variables for two components of the fluid rate of expansion tensor by

$$\alpha := (\tfrac{1}{3}\,\Theta - 2\,\sigma_+)\ , \qquad \beta := (\tfrac{1}{3}\,\Theta + \sigma_+)\ , \tag{4.7}$$

shadowing the variable names used earlier in a related context in Ref. [5] (on this choice of variables, see also the remarks made on the so-called "Taub gauge" for fluid spacetime geometries in Ref. [28]).

Substituting from Eq. (4.5) into the evolution equation for the semi-longitudinal fluid shear component σ_3 (see, e.g., Ref. [10]), one finds that the latter is *involutive*. Therefore, on performing a spatial rotation of $\{\,\mathbf{e}_\alpha\,\}$ about $\mathbf{e}_1$ at every point of a given 2-surfaces $\{t = \text{const}, x = \text{const}\}$, one can set

$$0 = \sigma_3 \tag{4.8}$$

to hold at every event of $(\mathcal{M},\,\mathbf{g},\,\mathbf{u})$.[9] Determining the evolution along $\mathbf{u}$ of the frame gauge source function $(T^0)^{-1}\,{T^0}_1 = \dot{u}_1$ as outlined in Subsec. II B, at this point, from a dynamical viewpoint, the coordinate and spatial frame freedom has been completely fixed (modulo coordinate reparameterisation freedom given by $t' = t'(t)$, $x' = x'(x)$, $y' = y + f(x)$ and $z' = z + g(x)$ [31]).

It is now fairly straightforward to derive from the equations of a $1+3$ orthonormal frame *connection* formulation of gravitational fields as given in Refs. [19] and [10] an evolution system of autonomous partial differential equations in FOSH format for the following set of eleven geometrically defined field variables $u^A = u^A(t,x)$:

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[8]For $0 = \omega^\alpha(\mathbf{u})$, the choice $0 = M_i$ is the simplest one possible, but it is by no means compulsory.

[9]Alternatively one could choose $\mathbf{e}_3 \parallel \boldsymbol{\eta}$, and then set $0 = \sigma_2$ instead.

$$u^A = \left(e_1{}^1,\, \beta,\, \sigma_2,\, a_1,\, \mu,\, \alpha,\, \dot{u}_1,\, \sigma_-,\, n_\times,\, \sigma_\times,\, n_- \right)^T \ . \tag{4.9}$$

Note that the evolution of the frame coordinate components other than $e_1{}^1$ is decoupled from this set; $e_1{}^1$ itself forms a background field for pairwise dynamical interactions between $\{\,\alpha,\, \dot{u}_1\,\}$, $\{\,\sigma_-,\, n_\times\,\}$ and $\{\,\sigma_\times,\, n_-\,\}$ in the sense of Geroch [15].

11-dimensional autonomous first-order symmetric hyperbolic evolution system:

$$M^{-1}\,\partial_t e_1{}^1 = -\,\alpha\, e_1{}^1 \tag{4.10}$$

$$M^{-1}\,\partial_t \beta = -\,\tfrac{3}{2}\,\beta^2 - \tfrac{3}{2}\,(\sigma_-^2 + \sigma_\times^2 + n_-^2 + n_\times^2) + \tfrac{3}{2}\,\sigma_2^2 - \tfrac{1}{2}\,(2\,\dot{u}_1 - a_1)\,a_1 - \tfrac{1}{2}\,(p - \Lambda) \tag{4.11}$$

$$M^{-1}\,\partial_t \sigma_2 = -\,(3\,\beta - \sqrt{3}\,\sigma_-)\,\sigma_2 \tag{4.12}$$

$$M^{-1}\,\partial_t a_1 = -\,\beta\,(\dot{u}_1 + a_1) + 3\,(n_-\,\sigma_\times - n_\times\,\sigma_-) \tag{4.13}$$

$$M^{-1}\,\partial_t \mu = -\,(\alpha + 2\,\beta)\,(\mu + p) \tag{4.14}$$

$$c_s^2\,M^{-1}\,\partial_t \alpha - c_s^2\,e_1{}^1\,\partial_x \dot{u}_1 = c_s^2\,[\,-\,\alpha^2 + \beta^2 - 3\,(\sigma_-^2 + \sigma_\times^2 - n_-^2 - n_\times^2) - 9\,\sigma_2^2 + \dot{u}_1^2 - a_1^2 - \tfrac{1}{2}\,(\mu + p)\,] \tag{4.15}$$

$$\begin{aligned} M^{-1}\,\partial_t \dot{u}_1 - c_s^2\ e_1{}^1\,\partial_x \alpha &= -\,\alpha\,\dot{u}_1 - (\alpha + 2\,\beta)\,[\,c_s^{-2}\,\frac{d^2p}{d\mu^2}\,(\mu + p) - c_s^2\,]\,\dot{u}_1 \\ &\quad - c_s^2\,[\,2\,a_1\,(\alpha - \beta) + 6\,(n_-\,\sigma_\times - n_\times\,\sigma_-)\,] \end{aligned} \tag{4.16}$$

$$M^{-1}\,\partial_t \sigma_- + e_1{}^1\,\partial_x n_\times = -\,(\alpha + 2\,\beta)\,\sigma_- + 2\sqrt{3}\,\sigma_\times^2 - \sqrt{3}\,\sigma_2^2 - 2\sqrt{3}\ n_-^2 - (\dot{u}_1 - 2\,a_1)\,n_\times \tag{4.17}$$

$$M^{-1}\,\partial_t n_\times + e_1{}^1\,\partial_x \sigma_- = -\,\alpha\,n_\times - \dot{u}_1\,\sigma_- \tag{4.18}$$

$$M^{-1}\,\partial_t \sigma_\times - e_1{}^1\,\partial_x n_- = -\,(\alpha + 2\,\beta + 2\sqrt{3}\,\sigma_-)\,\sigma_\times + (\dot{u}_1 - 2\,a_1 - 2\sqrt{3}\,n_\times)\,n_- \tag{4.19}$$

$$M^{-1}\,\partial_t n_- - e_1{}^1\,\partial_x \sigma_\times = -\,(\alpha - 2\sqrt{3}\,\sigma_-)\,n_- + (\dot{u}_1 + 2\sqrt{3}\,n_\times)\,\sigma_\times \ . \tag{4.20}$$

The characteristic condition for the set (4.10) – (4.20) is invariantly given by $0 = Q = (u^a\zeta_a)^5\,[\,(-\,u^b\,u^c + c_s^2\,h^{bc})\,\zeta_b\,\zeta_c\,]\,[\,(-\,u^d\,u^e + 1^2\,h^{de})\,\zeta_d\,\zeta_e\,]^2$, where $\zeta_a := \nabla_a\phi$ are the past-directed normals to characteristic 3-surfaces $\mathcal{C}$:$\{\phi(x^\mu) = \text{const}\}$. The *connection* characteristic eigenfields associated with *non-zero* characteristic velocities are $\{\,(\alpha \pm c_s\,\dot{u}_1)/(1 + c_s^2)^{1/2}\,\}$ with $v = \pm\,c_s$ and $\{\,(\sigma_- \mp n_\times)/\sqrt{2},\, (\sigma_\times \pm n_-)/\sqrt{2}\,\}$ with $v = \pm\,1$.

Initial data which is invariant under the transformations of an Abelian G_2 isometry group has to satisfy the following set of constraint equations:

Initial value constraint equations:

$$e_1{}^1\,M^{-1}\,\partial_x M = \dot{u}_1 \tag{4.21}$$

$$0 = (C_{\rm com})_1 \ := \ e_1{}^1\,\partial_x e_2{}^2 - (a_1 + \sqrt{3}\,n_\times)\,e_2{}^2 \tag{4.22}$$

$$0 = (C_{\rm com})_2 \ := \ e_1{}^1\,\partial_x e_3{}^2 - (a_1 - \sqrt{3}\,n_\times)\,e_3{}^2 + 2\sqrt{3}\,n_-\,e_2{}^2 \tag{4.23}$$

$$0 = (C_{\rm com})_3 \ := \ e_1{}^1\,\partial_x e_3{}^3 - (a_1 - \sqrt{3}\,n_\times)\,e_3{}^3 \tag{4.24}$$

$$0 = (C_{\rm G}) \ := \ {}^*\!R + 2\,(2\,\alpha + \beta)\,\beta - 6\,(\sigma_-^2 + \sigma_\times^2 + \sigma_2^2) - 2\,\mu - 2\,\Lambda \tag{4.25}$$

$$0 = (C_1)_1 \ := \ e_1{}^1\,\partial_x \beta + a_1\,(\alpha - \beta) + 3\,(n_-\,\sigma_\times - n_\times\,\sigma_-) \tag{4.26}$$

$$0 = (C_1)_3 \ := \ (e_1{}^1\,\partial_x - 3\,a_1 + \sqrt{3}\,n_\times)\,\sigma_2 \tag{4.27}$$

$$0 = (C_{\rm PF})_1 \ := \ c_s^2\,e_1{}^1\,\partial_x \mu + (\mu + p)\,\dot{u}_1 \ ; \tag{4.28}$$

the 3-Ricci curvature scalar ${}^*\!R$ of spacelike 3-surfaces $\mathcal{S}$:$\{t = \text{const}\}$ orthogonal to $\mathbf{u}$ is given by

$${}^*\!R := 2\,(2\,e_1{}^1\,\partial_x - 3\,a_1)\,a_1 - 6\,(n_-^2 + n_\times^2) \ . \tag{4.29}$$

The constraint equations (4.25) – (4.28) in this set are specialisations of Eqs. (A10), (A5), (A8) and (A15), respectively. The remaining ones, Eqs. (4.21) – (4.24), derive from the commutators.

Algorithm: Given an equation of state of the form $p = p(\mu)$, the *initial data*, which can be specified *freely* (modulo minimal differentiability requirements and the remaining coordinate reparameterisation freedom) as functions of the spatial coordinate x on a spacelike 3-surface $\mathcal{S}$:$\{t = \text{const}\}$ orthogonal to the fluid 4-velocity field $\mathbf{u}$, are the values of the variables $\{\,e_1{}^1,\, \alpha,\, \sigma_-,\, n_\times,\, \sigma_\times,\, n_-,\, \mu\,\}$, together with the cosmological constant Λ. Specifying the values of the variables $\{\,e_2{}^2,\, e_3{}^2,\, e_3{}^3,\, \beta,\, \sigma_2,\, a_1\,\}$ *at one point* on this 3-surface, their spatial distribution follows from Eqs. (4.22) – (4.25), respectively, while $\dot{u}_1$ is

determined through Eq. (4.28) and M through Eq. (4.21). Then all *time derivatives* of these variables are known. Note that one of the coordinate components ${e_1}^2$ and ${e_1}^3$ is arbitrary as functions of x, too, while the other follows from the unit-magnitude property of $\mathbf{e}_1$. It remains to specify *boundary conditions* for all variables to obtain unique solutions.

Two subcases of importance are contained within the class of Abelian G_2 perfect fluid cosmological models: the *orthogonally transitive* subcase arises when $0 = {e_1}^2 = {e_1}^3 \Rightarrow 0 = \sigma_2$, which itself specialises to the *diagonal* ("polarised") subcase when additionally $0 = {e_3}^2 \Rightarrow 0 = \sigma_\times = n_-$, leading to a diagonal line element (see Refs. [16] and [32]).

The initial value constraint equations (4.22) – (4.28) are propagated along $\mathbf{u}$ via a FOSH evolution system of autonomous partial differential equations, where the characteristic speeds are $|\,v\,| = 0$, according to

7-dimensional autonomous constraint evolution system:

$$M^{-1}\,\partial_t(C_{\rm com})_1 = -\,(\alpha+\beta+\sqrt{3}\,\sigma_-)\,(C_{\rm com})_1 - {e_2}^2\,(C_1)_1 \tag{4.30}$$
$$M^{-1}\,\partial_t(C_{\rm com})_2 = -\,(\alpha+\beta-\sqrt{3}\,\sigma_-)\,(C_{\rm com})_2 - 2\sqrt{3}\,\sigma_\times\,(C_{\rm com})_1 - {e_3}^2\,(C_1)_1 \tag{4.31}$$
$$M^{-1}\,\partial_t(C_{\rm com})_3 = -\,(\alpha+\beta-\sqrt{3}\,\sigma_-)\,(C_{\rm com})_3 - {e_3}^3\,(C_1)_1 \tag{4.32}$$
$$M^{-1}\,\partial_t(C_{\rm G}) = -\,(\alpha+\beta)\,(C_{\rm G}) - 4\,(\dot{u}_1+a_1)\,(C_1)_1 \tag{4.33}$$
$$M^{-1}\,\partial_t(C_1)_1 = -\,(\alpha+3\,\beta)\,(C_1)_1 + \sigma_2\,(C_1)_3 - \tfrac{1}{4}\,(\dot{u}_1-a_1)\,(C_{\rm G}) - \tfrac{1}{2}\,(C_{\rm PF})_1 \tag{4.34}$$
$$M^{-1}\,\partial_t(C_1)_3 = -\,(\alpha+3\,\beta-\sqrt{3}\,\sigma_-)\,(C_1)_3 - 9\,\sigma_2\,(C_1)_1 \tag{4.35}$$
$$M^{-1}\,\partial_t(C_{\rm PF})_1 = -\,2\,(\alpha+\beta)\,(C_{\rm PF})_1 - (\alpha+2\,\beta)\,[\,c_s^2 + c_s^{-2}\,\frac{d^2p}{d\mu^2}\,(\mu+p)\,]\,(C_{\rm PF})_1 - 2\,c_s^{-2}\,(\mu+p)\,(C_1)_1\ . \tag{4.36}$$

The full set of propagation equations for the Abelian G_2 perfect fluid cosmological models now forms a larger autonomous FOSH evolution system according to Eq. (2.1), with the previous 11-dimensional system as a subset. The virtue of the larger system is that it explicitly shows that the set of dynamical field equations introduced is consistent: the initial value constraint equations are preserved by the time evolution equations (if they are true initially, they remain true thereafter). This completes the discussion on a well-posed initial value problem for this class of cosmological models in the $1+3$ orthonormal frame *connection* formulation of gravitational fields.

We finally list the specialisations which the expressions (A18) – (A23) for the Weyl curvature characteristic eigenfields undergo by the geometrical restrictions imposed by the Abelian G_2 isometry group. We use Eq. (4.27) to eliminate derivatives $\partial_x\sigma_2$.

Weyl curvature characteristic eigenfields:

$$E_+ = -\,\tfrac{1}{3}\,{e_1}^1\,\partial_x a_1 - \tfrac{1}{3}\,(\alpha-\beta)\,\beta - \sigma_-^2 - \sigma_\times^2 + 2\,n_-^2 + 2\,n_\times^2 + \tfrac{1}{2}\,\sigma_2^2 \tag{4.37}$$
$$H_+ = -\,\tfrac{3}{2}\,(\sigma_- - n_\times)\,(\sigma_\times + n_-) + \tfrac{3}{2}\,(\sigma_- + n_\times)\,(\sigma_\times - n_-) \tag{4.38}$$
$$(E_3 \mp H_2) = -\,\sqrt{3}\,(\sigma_\times \pm n_-)\,\sigma_2 \tag{4.39}$$
$$(E_2 \pm H_3) = (\beta \mp a_1 + \sqrt{3}\,\sigma_- \mp \sqrt{3}\,n_\times)\,\sigma_2 \tag{4.40}$$
$$(E_- \mp H_\times) = \mp\,({e_1}^1\,\partial_x \mp \alpha - a_1)\,(\sigma_- \mp n_\times) \pm 2\sqrt{3}\,n_-\,(\sigma_\times \pm n_-) \pm n_\times\,(\beta \mp a_1) + \tfrac{\sqrt{3}}{2}\,\sigma_2^2 \tag{4.41}$$
$$(E_\times \pm H_-) = \mp\,({e_1}^1\,\partial_x \mp \alpha - a_1)\,(\sigma_\times \pm n_-) \mp 2\sqrt{3}\,n_-\,(\sigma_- \mp n_\times) \mp n_-\,(\beta \mp a_1)\ . \tag{4.42}$$

Note that $0 = \sigma_2 \Rightarrow 0 = (E_3 \mp H_2) = (E_2 \pm H_3)$ holds.

B. Transport equations for jump discontinuities in outward first derivatives

To give a graphic example, which, we believe, will also be of some interest in numerical investigations of dynamical features of Abelian G_2 perfect fluid cosmological models, we conclude this section with a brief derivation of the transport equations that describe how physically relevant jump discontinuities in the outward first derivatives of the initial data are propagated along the bicharacteristic rays of the setting. We confine ourselves to modes with $v \neq 0$ relative to $\mathbf{u}$. To this end, we first turn to Eq. (2.4) for each of $v \in \{\pm c_s, \pm 1\}$, leading to the conditions

$$[\partial_\phi \dot{u}_1] = \pm\,c_s\,[\partial_\phi\alpha]\ , \qquad [\partial_\phi n_\times] = \mp\,[\partial_\phi\sigma_-]\ , \qquad [\partial_\phi n_-] = \pm\,[\partial_\phi\sigma_\times]\ , \tag{4.43}$$

respectively. This then gives from Eq. (2.5), together with the evolution subsystem (4.15) – (4.20), the relations:

(i) $v = \pm c_s$ *longitudinal modes*:

$$0 = (M^{-1} \partial_t \mp c_s e_1{}^1 \partial_x + \tfrac{1}{2} c_s^{-1} (M^{-1} \partial_t c_s \mp c_s e_1{}^1 \partial_x c_s) + \alpha + \tfrac{1}{2} (\alpha + 2\beta) [c_s^{-2} \frac{d^2 p}{d\mu^2} (\mu + p) - c_s^2] \mp \tfrac{1}{2} c_s \dot{u}_1 \pm c_s a_1) [\partial_\phi(\alpha \pm c_s^{-1} \dot{u}_1)] \; ; \tag{4.44}$$

jump discontinuities in the outward first derivatives of initial data for $\{ \alpha, \dot{u}_1 \}$, subject to Eq. (4.43), travel along the local sound cones. Note that, because of the general functional dependence $c_s = c_s(\mu)$, the evolution of sound cone initial data typically leads to the formation of “shocks”. This phenomenon becomes impossible in the special case of a *linear* baryotropic equation of state with $p(\mu) = (\gamma - 1)\, \mu$, $1 \leq \gamma \leq 2$, where $c_s = (\gamma - 1)^{1/2} = \text{const}$.

(ii) $v = \pm 1$ *transverse modes*:

$$0 = (M^{-1} \partial_t \mp e_1{}^1 \partial_x + \alpha + \beta \mp \dot{u}_1 \pm a_1) [\partial_\phi(\sigma_- \mp n_\times)] \tag{4.45}$$

$$0 = (M^{-1} \partial_t \mp e_1{}^1 \partial_x + \alpha + \beta \mp \dot{u}_1 \pm a_1) [\partial_\phi(\sigma_\times \pm n_-)] \; . \tag{4.46}$$

jump discontinuities in the outward first derivatives of initial data for $\{ \sigma_-, n_\times \}$ and $\{ \sigma_\times, n_- \}$, subject to Eq. (4.43), travel along the local light cones.

V. CONCLUSION

The main conclusion of this work is that in examining a set of dynamical equations for a physical system such as the relativistic gravitational field equations, completed by the equations for all needed auxiliary variables, the constraint equations are crucial in determining what information can be propagated along the characteristic 3-surfaces of the evolution equations when these are expressed in FOSH format. We have explicitly shown how to examine the constraint equations to determine whether jump discontinuities in the derivatives of the initial data can be propagated along the various characteristic 3-surfaces in the case of the relativistic gravitational field equations with a baryotropic perfect fluid matter source, including the Weyl curvature variables. This makes it clear that such an investigation is needed to complement the determination of the set of characteristic 3-surfaces of any FOSH evolution system with any existing supplementary constraint equations, in order to determine which characteristic 3-surfaces in the set are physically relevant.

The process outlined enables us to show why the characteristic 3-surfaces for the semi-longitudinal Weyl curvature characteristic eigenfields $(E_3 \mp H_2)$ and $(E_2 \pm H_3)$ that are associated with $|\, v \,| = \frac{1}{2}$, apparent in a straightforward reduction of order $\partial^3 g$ of the evolution system of the relativistic gravitational field equations for a baryotropic perfect fluid matter source to FOSH format, cannot in fact be activated. It demonstrates that potentially associated semi-longitudinal gravitational radiation cannot occur, despite the occurrence of related characteristic 3-surfaces in the FOSH evolution system. It should be noted that the issue is *not* that the constraint equations are incompatible with the evolution equations, in the sense of not being conserved under the system's time evolution. On the contrary, the propagation of these constraint equations is indeed compatible with the existence of these modes, as can be shown by considering an extended FOSH evolution system that includes variables representing satisfaction of the constraint equations [18,13] (see also Refs. [29], [8] and [20]). The issue is that the constraint equations do not allow the setting of jump discontinuities in (the derivatives of) the initial data for $(E_3 \mp H_2)$ and $(E_2 \pm H_3)$ *because the values of the components of the constraint equations themselves cannot suffer jump discontinuities*: they have to be continuously zero from one spacetime event to any nearby one, i.e., everywhere.

What this analysis does not do is to show in what manner the semi-longitudinal Weyl curvature characteristic eigenfields $(E_3 \mp H_2)$ and $(E_2 \pm H_3)$, originally identified by Szekeres and shown there to have observable physical effects [26], will evolve in time, nor does it adequately characterise what freedom there is in setting initial data for these modes. It would be helpful to have some characterisation of the full freedom to assign these modes on an initial data 3-surface.

The worked example for perfect fluid cosmological models with an Abelian G_2 isometry group presented in Sec. IV features neatly the conceptual and mathematical advantages one can gain from combining the idea of FOSH evolution systems with a 1+3 orthonormal frame *connection* formulation of gravitational fields. It will be usable as a multi-facet test bed for numerical experiments of spacetime geometry evolution processes in relativistic cosmology and already provides the basis for work-in-progress on an interesting new scale-invariant, dimensionless dynamical formulation for the orthogonally transitive Abelian G_2 perfect fluid cosmological models [27].

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APPENDIX:

1. Suggestive example: linearised relativistic gravitational field equations in a metric approach with Regge–Wheeler coordinate gauge fixing

Rooted in tradition, a *metric approach* to formulating relativistic gravitational dynamics is still more frequently encountered in the literature than, e.g., the (extended) $1+3$ orthonormal frame formulation employed in this paper and in Ref. [9]. This motivates the brief discussion of a metric approach based example derived from a paper by Kind, Ehlers and Schmidt [17] that highlights the issue of the physical role of the constraint equations in the selection of appropriate initial data sets. The example arises as a special subcase of linearised relativistic gravitational field equations that describe small adiabatic, non-radial perturbations of a star in hydrostatic equilibrium. Starting from a set of local coordinates $\{\, t,\, r,\, \vartheta,\, \varphi \,\}$ and imposing Regge–Wheeler coordinate gauge fixing conditions, the Ansatz for the line element contains two real-valued metric functions $f = f(t,r)$ and $g = g(t,r)$; the angular behaviour of the line element can be given in terms of spherical harmonic functions $Y_{lm}(\vartheta, \varphi)$ (for more details see Sec. 2 of Ref. [17]). One can then obtain from the linearised Einstein field equations a *second-order* symmetric hyperbolic evolution system for f and g that takes the form

$$-\,\ddot{f} + f'' = \frac{2}{r}\,f' - \frac{4}{r^2}\,g' + \frac{l(l+1)}{r^2}\,f \tag{A1}$$

$$-\,\ddot{g} + g'' = -\,\frac{2}{r}\,g' + \frac{2}{r^2}\,f + \frac{1}{r^2}\,(\,l(l+1) - 2\,)\,g\ . \tag{A2}$$

Moreover, f and g are bound to satisfy the constraint equation

$$0 = (C) := g'' - \frac{1}{r}\,f' + \frac{3}{r}\,g' - \frac{l(l+1)}{2r^2}\,(f+g) - \frac{1}{r}\,(f-g)\ . \tag{A3}$$

The characteristic 3-surfaces $\mathcal{C}$ determined by the principal parts of Eqs. (A1) and (A2) satisfy the conditions $(t \mp r) = \text{const}$. If we assume that Eqs. (A1) and (A2) hold for f and g, then we obtain for (C) the evolution equation

$$0 = -\,(\ddot{C}) + (C)'' + \frac{2}{r}\,(C)' - \frac{l(l+1)}{2r^2}\,(C)\ . \tag{A4}$$

Hence, if on a spacelike 3-surface $\mathcal{S}$:$\{t = \text{const}\}$ real-valued initial data $I_{\partial^2 g} = \{\, f,\, \dot{f},\, g,\, \dot{g}\,\}$ satisfy $0 = (C) = (\dot{C})$, then any solution of Eqs. (A1) and (A2) satisfies $0 = (C)$ everywhere.

It is instructive to note that from Eq. (A3) we can now add any arbitrary real-valued multiple $a\,g''$ to Eq. (A2) to obtain a principal part of the form

$$-\,\ddot{g} + (1+a)\,g''\ ,$$

which instead defines characteristic 3-surfaces $\mathcal{C}$ that satisfy the condition $(\,(1+a)^{1/2}\,t \mp r\,) = \text{const}$. In particular, we can choose $a = -\,1$, leading to $r = \text{const}$, which corresponds to a propagation speed $|\,v\,| = 0$. This implies that no information is transported from one event to a spatially separated nearby one. A different view considers the differentiability properties of f and g: if one chooses real-valued initial data such that $\{\, f,\, f',\, g,\, g'\,\}$ are *continuous*, then the constraint equation (A3) imposes *no condition* on the differentiability of f'' (and so on $\ddot{f}$, from Eq. (A1)), while g'' (and so $\ddot{g}$, from Eq. (A2)) is required to be *continuous* too. Hence, jump discontinuities are physically allowed only in the initial value of the former quantity and are propagated via Eq. (A1).

2. Constraint equations component-wise

Fluid shear divergence equations/Jacobi constraint equations:

$$\begin{aligned}0 = -\tfrac{1}{2}\,(C_1)_1 &= \mathbf{e}_1(\tfrac{1}{3}\,\Theta + \sigma_+) - \tfrac{\sqrt{3}}{2}\,(\mathbf{e}_2 - 3\,a_2 + \sqrt{3}\,n_2)\,(\sigma_3) - \tfrac{\sqrt{3}}{2}\,(\mathbf{e}_3 - 3\,a_3 - \sqrt{3}\,n_3)\,(\sigma_2) \\ &\quad - \tfrac{1}{2}\,(\mathbf{e}_2 + 2\,\dot{u}_2 - a_2 - \sqrt{3}\,n_2)\,(\omega_3) + \tfrac{1}{2}\,(\mathbf{e}_3 + 2\,\dot{u}_3 - a_3 + \sqrt{3}\,n_3)\,(\omega_2) \\ &\quad - 3\,a_1\,\sigma_+ + 3\,(n_-\,\sigma_\times - n_\times\,\sigma_-) + \tfrac{1}{6}\,(n - 2\,n_+)\,\omega_1 \end{aligned} \tag{A5}$$

$$\begin{aligned}0 = (C_{\rm J})_1 &= \tfrac{1}{3}\,(\mathbf{e}_1 - 2\,a_1)\,(n - 2n_+) + (\mathbf{e}_2 - 2\,a_2)\,(a_3 + \sqrt{3}n_3) + (\mathbf{e}_3 - 2\,a_3)\,(a_2 - \sqrt{3}n_2) \\ &\quad + 2\,(\tfrac{1}{3}\,\Theta - 2\,\sigma_+)\,\omega_1 + 2\,(\sqrt{3}\sigma_2 + \Omega_2)\,\omega_3 + 2\,(\sqrt{3}\sigma_3 - \Omega_3)\,\omega_2 \end{aligned} \tag{A6}$$

$$\begin{aligned}0 = \tfrac{1}{\sqrt{3}}\,[\,(C_1)_2 \mp (C_{\rm J})_3\,] &= \mathbf{e}_1(\sigma_3 \mp n_2 - \tfrac{1}{\sqrt{3}}\,\omega_3 \mp \tfrac{1}{\sqrt{3}}\,a_2) + \mathbf{e}_2(\sigma_- \mp n_\times \pm \tfrac{1}{\sqrt{3}}\,a_1) + \mathbf{e}_3(\sigma_\times \pm n_- + \tfrac{1}{\sqrt{3}}\,\omega_1) \\ &\quad + \tfrac{1}{\sqrt{3}}\,(\mathbf{e}_2 - 3\,a_2 + 3\sqrt{3}\,n_2)\,(\sigma_+) \mp \tfrac{1}{3\sqrt{3}}\,(\mathbf{e}_3 - 2\,a_3)\,(n + n_+) - \tfrac{2}{3\sqrt{3}}\,\mathbf{e}_2(\Theta) \\ &\quad - 3\,a_1\,(\sigma_3 \mp n_2) - 3\,a_2\,(\sigma_- \mp n_\times) - 3\,a_3\,(\sigma_\times \pm n_-) - (n_+ - \sqrt{3}\,n_-)\,\sigma_2 \\ &\quad - n_\times\,(\sqrt{3}\,\sigma_3 \pm a_2) - n_2\,(\sqrt{3}\,\sigma_- \pm a_1) + \sqrt{3}\,n_3\,\sigma_\times \pm a_3\,n_- \\ &\quad \mp \tfrac{1}{\sqrt{3}}\,(2\sqrt{3}\,\sigma_2 \mp 2\,\dot{u}_3 - 2\,\Omega_2 \pm a_3 \pm \sqrt{3}\,n_3)\,\omega_1 \mp \tfrac{1}{\sqrt{3}}\,(2\sqrt{3}\,\sigma_\times + 2\,\Omega_1 \pm \tfrac{1}{3}\,n \pm \tfrac{1}{3}\,n_+ \pm \sqrt{3}\,n_-)\,\omega_2 \\ &\quad \mp \tfrac{1}{\sqrt{3}}\,(\tfrac{2}{3}\,\Theta + 2\,\sigma_+ - 2\sqrt{3}\,\sigma_- \pm 2\,\dot{u}_1 \mp a_1 \pm \sqrt{3}\,n_\times)\,\omega_3 \end{aligned} \tag{A7}$$

$$\begin{aligned}0 = \tfrac{1}{\sqrt{3}}\,[\,(C_1)_3 \pm (C_{\rm J})_2\,] &= \mathbf{e}_1(\sigma_2 \pm n_3 + \tfrac{1}{\sqrt{3}}\,\omega_2 \mp \tfrac{1}{\sqrt{3}}\,a_3) + \mathbf{e}_2(\sigma_\times \pm n_- - \tfrac{1}{\sqrt{3}}\,\omega_1) - \mathbf{e}_3(\sigma_- \mp n_\times \mp \tfrac{1}{\sqrt{3}}\,a_1) \\ &\quad \pm \tfrac{1}{3\sqrt{3}}\,(\mathbf{e}_2 - 2\,a_2)\,(n + n_+) + \tfrac{1}{\sqrt{3}}\,(\mathbf{e}_3 - 3\,a_3 - 3\sqrt{3}\,n_3)\,(\sigma_+) - \tfrac{2}{3\sqrt{3}}\,\mathbf{e}_3(\Theta) \\ &\quad - 3\,a_1\,(\sigma_2 \pm n_3) - 3\,a_2\,(\sigma_\times \pm n_-) + 3\,a_3\,(\sigma_- \mp n_\times) + (n_+ + \sqrt{3}\,n_-)\,\sigma_3 \\ &\quad + n_\times\,(\sqrt{3}\,\sigma_2 \pm a_3) - n_3\,(\sqrt{3}\,\sigma_- \mp a_1) - \sqrt{3}\,n_2\,\sigma_\times \pm a_2\,n_- \\ &\quad \pm \tfrac{1}{\sqrt{3}}\,(2\sqrt{3}\,\sigma_3 \mp 2\,\dot{u}_2 + 2\,\Omega_3 \pm a_2 \mp \sqrt{3}\,n_2)\,\omega_1 \pm \tfrac{1}{\sqrt{3}}\,(\tfrac{2}{3}\,\Theta + 2\,\sigma_+ + 2\sqrt{3}\,\sigma_- \pm 2\,\dot{u}_1 \mp a_1 \mp \sqrt{3}\,n_\times)\,\omega_2 \\ &\quad \pm \tfrac{1}{\sqrt{3}}\,(2\sqrt{3}\,\sigma_\times - 2\,\Omega_1 \mp \tfrac{1}{3}\,n \mp \tfrac{1}{3}\,n_+ \pm \sqrt{3}\,n_-)\,\omega_3\ . \end{aligned} \tag{A8}$$

Fluid vorticity divergence equation:

$$0 = (C_2) = (\mathbf{e}_1 - \dot{u}_1 - 2\,a_1)\,(\omega_1) + (\mathbf{e}_2 - \dot{u}_2 - 2\,a_2)\,(\omega_2) + (\mathbf{e}_3 - \dot{u}_3 - 2\,a_3)\,(\omega_3)\ . \tag{A9}$$

Generalised Gauß–Friedmann equation:

$$\begin{aligned}0 = (C_{\rm G}) &= 2\,(2\,\mathbf{e}_1 - 3\,a_1)\,(a_1) + 2\,(2\,\mathbf{e}_2 - 3\,a_2)\,(a_2) + 2\,(2\,\mathbf{e}_3 - 3\,a_3)\,(a_3) \\ &\quad + \tfrac{1}{6}\,n^2 - \tfrac{2}{3}\,n_+^2 - 6\,(n_-^2 + n_\times^2 + n_2^2 + n_3^2) \\ &\quad + 6\,(\tfrac{1}{3}\,\Theta - \sigma_+)\,(\tfrac{1}{3}\,\Theta + \sigma_+) - 6\,(\sigma_-^2 + \sigma_\times^2 + \sigma_2^2 + \sigma_3^2) \\ &\quad + 2\,(\omega_1 - 2\,\Omega_1)\,\omega_1 + 2\,(\omega_2 - 2\,\Omega_2)\,\omega_2 + 2\,(\omega_3 - 2\,\Omega_3)\,\omega_3 - 2\,\mu - 2\,\Lambda\ . \end{aligned} \tag{A10}$$

Weyl curvature divergence equations:

$$\begin{aligned}0 = -\tfrac{1}{2}\,(C_4)_1 &= (\mathbf{e}_1 - 3\,a_1)\,(E_+) - \tfrac{\sqrt{3}}{2}\,(\mathbf{e}_2 - 3\,a_2 + \sqrt{3}\,n_2)\,(E_3) - \tfrac{\sqrt{3}}{2}\,(\mathbf{e}_3 - 3\,a_3 - \sqrt{3}\,n_3)\,(E_2) \\ &\quad + \tfrac{1}{6}\,\mathbf{e}_1(\mu) + 3\,n_-\,E_\times - 3\,n_\times\,E_- + 3\,\sigma_-\,H_\times - 3\,\sigma_\times\,H_- \\ &\quad - 3\,\omega_1\,H_+ - \tfrac{3}{2}\,(\sigma_2 - \sqrt{3}\,\omega_2)\,H_3 + \tfrac{3}{2}\,(\sigma_3 + \sqrt{3}\,\omega_3)\,H_2 \end{aligned} \tag{A11}$$

$$\begin{aligned}0 = -\tfrac{1}{2}\,(C_5)_1 &= (\mathbf{e}_1 - 3\,a_1)\,(H_+) - \tfrac{\sqrt{3}}{2}\,(\mathbf{e}_2 - 3\,a_2 + \sqrt{3}\,n_2)\,(H_3) - \tfrac{\sqrt{3}}{2}\,(\mathbf{e}_3 - 3\,a_3 - \sqrt{3}\,n_3)\,(H_2) \\ &\quad - \tfrac{1}{2}\,(\mu + p)\,\omega_1 + 3\,n_-\,H_\times - 3\,n_\times\,H_- - 3\,\sigma_-\,E_\times + 3\,\sigma_\times\,E_- \\ &\quad + 3\,\omega_1\,E_+ + \tfrac{3}{2}\,(\sigma_2 - \sqrt{3}\,\omega_2)\,E_3 - \tfrac{3}{2}\,(\sigma_3 + \sqrt{3}\,\omega_3)\,E_2 \end{aligned} \tag{A12}$$

$$
\begin{aligned}
0 = \tfrac{1}{\sqrt{3}}\,[\,(C_4)_2 \mp (C_5)_3\,] &= (\mathbf{e}_1 \pm 3\,\sigma_+ - 3\,a_1)\,(E_3 \mp H_2) + (\mathbf{e}_2 \mp \sqrt{3}\,\sigma_3 \pm 3\,\omega_3 - 3\,a_2 - \sqrt{3}\,n_2)\,(E_- \mp H_\times) \\
&+ (\mathbf{e}_3 \mp \sqrt{3}\,\sigma_2 \mp 3\,\omega_2 - 3\,a_3 + \sqrt{3}\,n_3)\,(E_\times \pm H_-) + \tfrac{1}{\sqrt{3}}\,(\mathbf{e}_2 \mp 3\sqrt{3}\,\sigma_3 \mp 3\,\omega_3 - 3\,a_2 + 3\sqrt{3}\,n_2)\,(E_+) \\
&\mp \tfrac{1}{\sqrt{3}}\,(\mathbf{e}_3 \mp 3\sqrt{3}\,\sigma_2 \pm 3\,\omega_2 - 3\,a_3 - 3\sqrt{3}\,n_3)\,(H_+) - \tfrac{1}{3\sqrt{3}}\,\mathbf{e}_2(\mu) \mp \tfrac{1}{\sqrt{3}}\,(\mu+p)\,\omega_3 \\
&\pm \sqrt{3}\,(\sigma_- \mp n_\times)\,(E_3 \pm H_2) \pm \sqrt{3}\,(\sigma_\times \pm n_-)\,(E_2 \mp H_3) \mp (3\,\omega_1 \pm n_+)\,(E_2 \pm H_3)
\end{aligned}
\tag{A13}
$$

$$
\begin{aligned}
0 = \tfrac{1}{\sqrt{3}}\,[\,(C_4)_3 \pm (C_5)_2\,] &= (\mathbf{e}_1 \pm 3\,\sigma_+ - 3\,a_1)\,(E_2 \pm H_3) + (\mathbf{e}_2 \mp \sqrt{3}\,\sigma_3 \pm 3\,\omega_3 - 3\,a_2 - \sqrt{3}\,n_2)\,(E_\times \pm H_-) \\
&- (\mathbf{e}_3 \mp \sqrt{3}\,\sigma_2 \mp 3\,\omega_2 - 3\,a_3 + \sqrt{3}\,n_3)\,(E_- \mp H_\times) \pm \tfrac{1}{\sqrt{3}}\,(\mathbf{e}_2 \mp 3\sqrt{3}\,\sigma_3 \mp 3\,\omega_3 - 3\,a_2 + 3\sqrt{3}\,n_2)\,(H_+) \\
&+ \tfrac{1}{\sqrt{3}}\,(\mathbf{e}_3 \mp 3\sqrt{3}\,\sigma_2 \pm 3\,\omega_2 - 3\,a_3 - 3\sqrt{3}\,n_3)\,(E_+) - \tfrac{1}{3\sqrt{3}}\,\mathbf{e}_3(\mu) \pm \tfrac{1}{\sqrt{3}}\,(\mu+p)\,\omega_2 \\
&\mp \sqrt{3}\,(\sigma_- \mp n_\times)\,(E_2 \mp H_3) \pm \sqrt{3}\,(\sigma_\times \pm n_-)\,(E_3 \pm H_2) \pm (3\,\omega_1 \pm n_+)\,(E_3 \mp H_2)\ .
\end{aligned}
\tag{A14}
$$

Momentum conservation equations:

$$0 = (C_{\rm PF})_1 = c_s^2\,\mathbf{e}_1(\mu) + (\mu+p)\,\dot{u}_1 \tag{A15}$$
$$0 = (C_{\rm PF})_2 = c_s^2\,\mathbf{e}_2(\mu) + (\mu+p)\,\dot{u}_2 \tag{A16}$$
$$0 = (C_{\rm PF})_3 = c_s^2\,\mathbf{e}_3(\mu) + (\mu+p)\,\dot{u}_3\ . \tag{A17}$$

Weyl curvature characteristic eigenfields:

$$
\begin{aligned}
E_+ = &- \tfrac{1}{3}\,\mathbf{e}_1(a_1) + \tfrac{1}{6}\,(\mathbf{e}_2 - 3\sqrt{3}\,n_2)\,(a_2) + \tfrac{1}{6}\,(\mathbf{e}_3 + 3\sqrt{3}\,n_3)\,(a_3) \\
&+ \tfrac{\sqrt{3}}{2}\,(\mathbf{e}_2 - a_2 - \tfrac{1}{\sqrt{3}}\,n_2)\,(n_2) - \tfrac{\sqrt{3}}{2}\,(\mathbf{e}_3 - a_3 + \tfrac{1}{\sqrt{3}}\,n_3)\,(n_3) \\
&+ (\tfrac{1}{3}\,\Theta + \sigma_+)\,\sigma_+ + \tfrac{1}{3}\,(n - 2\,n_+)\,n_+ \\
&- (\sigma_- - n_\times)\,(\sigma_- + n_\times) - (\sigma_\times + n_-)\,(\sigma_\times - n_-) \\
&+ \tfrac{1}{2}\,(\sigma_3 - n_2)\,(\sigma_3 + n_2) + \tfrac{1}{2}\,(\sigma_2 + n_3)\,(\sigma_2 - n_3) \\
&+ n_-^2 + n_\times^2 + \tfrac{1}{3}\,(\omega_1 - 2\,\Omega_1)\,\omega_1 - \tfrac{1}{6}\,(\omega_2 - 2\,\Omega_2)\,\omega_2 - \tfrac{1}{6}\,(\omega_3 - 2\,\Omega_3)\,\omega_3 \\
&+ \tfrac{1}{2}\,(C_{\rm G})_{11}
\end{aligned}
\tag{A18}
$$

$$
\begin{aligned}
H_+ = &- \tfrac{\sqrt{3}}{2}\,(\mathbf{e}_2 - a_2 - \sqrt{3}\,n_2)\,(\sigma_2) + \tfrac{\sqrt{3}}{2}\,(\mathbf{e}_3 - a_3 + \sqrt{3}\,n_3)\,(\sigma_3) \\
&+ \tfrac{1}{3}\,(\mathbf{e}_1 + 2\,\dot{u}_1 + a_1)\,(\omega_1) - \tfrac{1}{6}\,(\mathbf{e}_2 + 2\,\dot{u}_2 + a_2 - 3\sqrt{3}\,n_2)\,(\omega_2) \\
&- \tfrac{1}{6}\,(\mathbf{e}_3 + 2\,\dot{u}_3 + a_3 + 3\sqrt{3}\,n_3)\,(\omega_3) \\
&- \tfrac{1}{2}\,(n - 2\,n_+)\,\sigma_+ - \tfrac{3}{2}\,(\sigma_- - n_\times)\,(\sigma_\times + n_-) + \tfrac{3}{2}\,(\sigma_- + n_\times)\,(\sigma_\times - n_-) \\
&- \tfrac{1}{2}\,(C_3)_{11}
\end{aligned}
\tag{A19}
$$

$$
\begin{aligned}
(E_3 \mp H_2) = &\mp \tfrac{1}{2}\,\mathbf{e}_1(\sigma_3 \mp n_2 - \tfrac{1}{\sqrt{3}}\,\omega_3 \mp \tfrac{1}{\sqrt{3}}\,a_2) \pm \tfrac{1}{2}\,\mathbf{e}_2(\sigma_- \mp n_\times \pm \tfrac{1}{\sqrt{3}}\,a_1) \\
&\pm \tfrac{1}{2}\,\mathbf{e}_3(\sigma_\times \pm n_- + \tfrac{1}{\sqrt{3}}\,\omega_1) \mp \tfrac{\sqrt{3}}{2}\,(\mathbf{e}_2 - a_2 + \sqrt{3}\,n_2)\,(\sigma_+) + \tfrac{1}{2\sqrt{3}}\,(\mathbf{e}_3 - 2\,a_3)\,(n_+) \\
&- \sqrt{3}\,(\sigma_2 \pm \tfrac{1}{2\sqrt{3}}\,a_3 \mp \tfrac{3}{2}\,n_3)\,(\sigma_\times \pm n_-) - \sqrt{3}\,(\sigma_3 \pm \tfrac{1}{2\sqrt{3}}\,a_2 \pm \tfrac{3}{2}\,n_2)\,(\sigma_- \mp n_\times) \\
&+ (\tfrac{1}{3}\,\Theta + \sigma_+)\,\sigma_3 \pm \tfrac{1}{2}\,a_1\,(\sigma_3 \mp 2\,n_2) \pm \tfrac{1}{2}\,(n - n_+)\,\sigma_2 + \tfrac{1}{3}\,(n - 2\,n_+)\,n_3 \\
&\mp \tfrac{\sqrt{3}}{2}\,n_-\,(\sigma_2 \mp n_3 \pm \tfrac{1}{\sqrt{3}}\,a_3) \pm \tfrac{\sqrt{3}}{2}\,n_\times\,(\sigma_3 \pm n_2 \pm \tfrac{1}{\sqrt{3}}\,a_2) \\
&\pm \tfrac{1}{2\sqrt{3}}\,(2\,\dot{u}_3 \mp \omega_2 \pm 2\,\Omega_2 + a_3 + \sqrt{3}\,n_3)\,\omega_1 \\
&- \tfrac{1}{2\sqrt{3}}\,(\omega_1 - 2\,\Omega_1 \mp n_+ \pm \sqrt{3}\,n_-)\,\omega_2 \pm \tfrac{1}{2\sqrt{3}}\,(2\,\dot{u}_1 + a_1 - \sqrt{3}\,n_\times)\,\omega_3 \\
&- \tfrac{1}{\sqrt{3}}\,[\,(C_{\rm G})_{12} \pm (C_3)_{31}\,]
\end{aligned}
\tag{A20}
$$

$$
\begin{aligned}
(E_2 \pm H_3) = &\mp \tfrac{1}{2}\,\mathbf{e}_1(\sigma_2 \pm n_3 + \tfrac{1}{\sqrt{3}}\,\omega_2 \mp \tfrac{1}{\sqrt{3}}\,a_3) \pm \tfrac{1}{2}\,\mathbf{e}_2(\sigma_\times \pm n_- - \tfrac{1}{\sqrt{3}}\,\omega_1) \\
&\mp \tfrac{1}{2}\,\mathbf{e}_3(\sigma_- \mp n_\times \mp \tfrac{1}{\sqrt{3}}\,a_1) - \tfrac{1}{2\sqrt{3}}\,(\mathbf{e}_2 - 2\,a_2)\,(n_+) \mp \tfrac{\sqrt{3}}{2}\,(\mathbf{e}_3 - a_3 - \sqrt{3}\,n_3)\,(\sigma_+)
\end{aligned}
$$

$$
\begin{aligned}
&+ \sqrt{3}\,(\sigma_2 \pm \tfrac{1}{2\sqrt{3}}\,a_3 \mp \tfrac{3}{2}\,n_3)\,(\sigma_- \mp n_\times) - \sqrt{3}\,(\sigma_3 \pm \tfrac{1}{2\sqrt{3}}\,a_2 \pm \tfrac{3}{2}\,n_2)\,(\sigma_\times \pm n_-) \\
&+ (\tfrac{1}{3}\,\Theta + \sigma_+)\,\sigma_2 \pm \tfrac{1}{2}\,a_1\,(\sigma_2 \pm 2\,n_3) \mp \tfrac{1}{2}\,(n - n_+)\,\sigma_3 + \tfrac{1}{3}\,(n - 2\,n_+)\,n_2 \\
&\mp \tfrac{\sqrt{3}}{2}\,n_-\,(\sigma_3 \pm n_2 \pm \tfrac{1}{\sqrt{3}}\,a_2) \mp \tfrac{\sqrt{3}}{2}\,n_\times\,(\sigma_2 \mp n_3 \pm \tfrac{1}{\sqrt{3}}\,a_3) \\
&\mp \tfrac{1}{2\sqrt{3}}\,(2\,\dot{u}_2 \pm \omega_3 \mp 2\,\Omega_3 + a_2 - \sqrt{3}\,n_2)\,\omega_1 \\
&\mp \tfrac{1}{2\sqrt{3}}\,(2\,\dot{u}_1 + a_1 + \sqrt{3}\,n_\times)\,\omega_2 - \tfrac{1}{2\sqrt{3}}\,(\omega_1 - 2\,\Omega_1 \mp n_+ \mp \sqrt{3}\,n_-)\,\omega_3 \\
&- \tfrac{1}{\sqrt{3}}\,[\,(C_{\mathrm{G}})_{31} \mp (C_3)_{12}\,] \qquad \text{(A21)}
\end{aligned}
$$

$$
\begin{aligned}
(E_- \mp H_\times) = &\mp \mathbf{e}_1(\sigma_- \mp n_\times) \pm \tfrac{1}{2}\,\mathbf{e}_2(\sigma_3 \mp n_2 + \tfrac{1}{\sqrt{3}}\,\omega_3 \pm \tfrac{1}{\sqrt{3}}\,a_2) \mp \tfrac{1}{2}\,\mathbf{e}_3(\sigma_2 \pm n_3 - \tfrac{1}{\sqrt{3}}\,\omega_2 \pm \tfrac{1}{\sqrt{3}}\,a_3) \\
&+ \tfrac{\sqrt{3}}{2}\,(\sigma_2 \pm \tfrac{1}{\sqrt{3}}\,a_3 \pm 2\,n_3)\,(\sigma_2 \pm n_3) - \tfrac{\sqrt{3}}{2}\,(\sigma_3 \pm \tfrac{1}{\sqrt{3}}\,a_2 \mp 2\,n_2)\,(\sigma_3 \mp n_2) \\
&+ (\tfrac{1}{3}\,\Theta - 2\,\sigma_+)\,\sigma_- \pm 3\,n_\times\,\sigma_+ \pm \tfrac{1}{2}\,(n + 2\,n_+)\,\sigma_\times + \tfrac{1}{3}\,(n + 4\,n_+)\,n_- \\
&\pm a_1\,(\sigma_- \mp 2\,n_\times) + \tfrac{1}{2}\,a_2\,n_2 + \tfrac{1}{2}\,a_3\,n_3 \pm n_-\,\omega_1 \\
&\pm \tfrac{1}{2\sqrt{3}}\,(2\,\dot{u}_3 \mp \omega_2 \pm 2\,\Omega_2 + a_3 - \sqrt{3}\,n_3)\,\omega_2 \pm \tfrac{1}{2\sqrt{3}}\,(2\,\dot{u}_2 \pm \omega_3 \mp 2\,\Omega_3 + a_2 + \sqrt{3}\,n_2)\,\omega_3 \\
&- \tfrac{1}{2\sqrt{3}}\,[\,(C_{\mathrm{G}})_{22} - (C_{\mathrm{G}})_{33} \pm 2\,(C_3)_{23}\,] \qquad \text{(A22)}
\end{aligned}
$$

$$
\begin{aligned}
(E_\times \pm H_-) = &\mp \mathbf{e}_1(\sigma_\times \pm n_-) \pm \tfrac{1}{2}\,\mathbf{e}_2(\sigma_2 \pm n_3 - \tfrac{1}{\sqrt{3}}\,\omega_2 \pm \tfrac{1}{\sqrt{3}}\,a_3) \pm \tfrac{1}{2}\,\mathbf{e}_3(\sigma_3 \mp n_2 + \tfrac{1}{\sqrt{3}}\,\omega_3 \pm \tfrac{1}{\sqrt{3}}\,a_2) \\
&- \tfrac{\sqrt{3}}{2}\,(\sigma_2 \pm \tfrac{1}{\sqrt{3}}\,a_3 \pm 2\,n_3)\,(\sigma_3 \mp n_2) - \tfrac{\sqrt{3}}{2}\,(\sigma_3 \pm \tfrac{1}{\sqrt{3}}\,a_2 \mp 2\,n_2)\,(\sigma_2 \pm n_3) \\
&+ (\tfrac{1}{3}\,\Theta - 2\,\sigma_+)\,\sigma_\times \mp 3\,n_-\,\sigma_+ \mp \tfrac{1}{2}\,(n + 2\,n_+)\,\sigma_- + \tfrac{1}{3}\,(n + 4\,n_+)\,n_\times \\
&\pm a_1\,(\sigma_\times \pm 2\,n_-) - \tfrac{1}{2}\,a_2\,n_3 + \tfrac{1}{2}\,a_3\,n_2 \pm n_\times\,\omega_1 \\
&\mp \tfrac{1}{2\sqrt{3}}\,(2\,\dot{u}_2 \pm \omega_3 \mp 2\,\Omega_3 + a_2 + \sqrt{3}\,n_2)\,\omega_2 \pm \tfrac{1}{2\sqrt{3}}\,(2\,\dot{u}_3 \mp \omega_2 \pm 2\,\Omega_2 + a_3 - \sqrt{3}\,n_3)\,\omega_3 \\
&- \tfrac{1}{2\sqrt{3}}\,[\,2\,(C_{\mathrm{G}})_{23} \mp (C_3)_{22} \pm (C_3)_{33}\,]\;. \qquad \text{(A23)}
\end{aligned}
$$

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TABLE I. Conventions for $(1+1+2)$–decomposition

Outward frame derivative (across $\mathcal{J}$):	$\mathbf{e}_1$
Tangential frame derivatives (along $\mathcal{J}$):	$\mathbf{e}_2$, $\mathbf{e}_3$
(Semi-)*Longitudinal* tensor/connection frame components (wrt. $\mathbf{e}_1$):	a_+, a_2, a_3 / n, n_+, n_2, n_3
Transverse tensor/connection frame components (wrt. $\mathbf{e}_1$):	a_-, $a_\times$ / n_-, $n_\times$